Phys 210B — Winter 2024

Problem Set #5

This is the final problem set; you are to work alone except for consultation with class notes and with the instructor. (No book, no internet.) Please bring your solution to Urey 7222 by 5pm on Wednesday March 20th.

1. **Conservative interface dynamics** — For the interface problem described by the Hamiltonian $\mathcal{H} = \frac{\sigma}{2}\int d^d\vec{x}\,(\vec{\nabla}h)^2$, we will consider the case of *conservative* dynamics, i.e., $\int d^d\vec{x}\,h(\vec{x}) =$ const. except at the boundaries.

   (a) Write down a Langevin equation of motion for $h(\vec{x},t)$, starting from an expression for the local transport current $\vec{J}(h)$ and assuming noisy gradient descent dynamics. Write down the first two moments of the effective Langevin noise $\eta(\vec{x},t)$ that enters additively into the equation for $h(\vec{x},t)$.

   (b) Find an expression for the height-height correlator $\widehat{C}(\vec{k},t)$ in this case, for $h(\vec{x},t=0)=0$. Show that in the long-time limit, the correlator $\widehat{C}^*(\vec{k}) = \widehat{C}(\vec{k},t\to\infty)$ is the same as the one we worked out in class for (non-conservative) relaxational dynamics. Explain why this must be the case.

   (c) In analogy with what we discussed in class, the interfacial fluctuation, as characterized by $\widetilde{C}(\vec{x}-\vec{x}',t) = \frac{1}{2}\langle[h(\vec{x},t)-h(\vec{x}',t)]^2\rangle$, can be written in the *scaling form*

   $$\widetilde{C}(\vec{r},t) = \frac{D}{\gamma}\,r^{2\alpha}g(t/r^z).$$

   Work out the scaling exponents α and z for this case.

   (d) Suppose the system is driven out of equilibrium by a force that drives the interfacial particles in a direction $\vec{a}$ in the "plane" of the interface. Write down examples of nonlinear terms allowed by the non-equilibrium dynamics. [Note that the dynamics is still conservative.] For the terms that are not allowed, explain why they are not allowed. What are the leading order nonlinearities in the equation of motion assuming that $|\vec{\nabla}h|$ is small?

2. **Nonlinear interface growth** — As discussed in class, the stochastic dynamics of a growing interface can be described by the KPZ equation,

   $$\frac{\partial h}{\partial t} = \gamma\nabla^2 h + \frac{\lambda}{2}\left(\nabla h\right)^2 + \eta(\vec{x},t),$$

   with $\langle\eta\rangle = 0$ and $\langle\eta(\vec{x},t)\eta(\vec{x}',t')\rangle = 2D\delta^d(\vec{x}-\vec{x}')\delta(t-t')$.

   (a) From the integral equation for the Fourier Transform $\widehat{h}(\vec{k},\omega) = \int d^d\vec{x}\,dt\,e^{i\vec{k}\cdot\vec{x}-i\omega t}h(\vec{x},t)$ derived in class, write down the perturbative solution, $\widehat{h}_2(\vec{k},\omega)$, to the second order in λ. Express the result in terms of $\widehat{G}_0(\vec{k},\omega) = 1/(\gamma k^2 - i\omega)$, $\widehat{\eta}(\vec{k},\omega)$, and their integrals.

   (b) Show that $\langle\partial h_2/\partial t\rangle$ is a constant and express the result in term of a dimensionless integral over $\vec{k}$. What is the physical meaning of this?

(c) In 1-dimension, the Fourier Transform of the correlation function $C(x - x', t - t') = \langle h(x,t)h(x',t')\rangle$, $\widehat{C}(k,\omega)$, is given by

$$\widehat{C}(k,\omega) = \frac{2\widehat{D}(k)}{(\widehat{\gamma}(k)k^2)^2 + \omega^2},$$

where $\widehat{D}(k) \propto \widehat{\gamma}(k) \propto k^{-1/2}$. Show the time-dependent correlation function has the scaling form

$$C(r,\tau) = |r|^{2\alpha} g(\tau/|r|^z),$$

and describe its behavior in the large-r and large-τ regimes. Find the exponents α and z and comment on the meaning of the exponent values obtained.