Phys 210B — Winter 2024

Problem Set #3: Fokker-Planck equation and Multiplicative Noise

Due date: Monday Feb 26

1. **Multiplicative noise** — The Langevin equation

$$\frac{dW}{dt} = \alpha W(t) + \eta(t) \cdot W(t) + 1 \tag{1}$$

with constant α and Gaussian noise ($\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = 2D\delta(t-t')$) describes many interesting phenomena. For example, $W(t)$ can be interpreted as the accumulated return on investment made via a steady (e.g., monthly) deposit. In this problem, you will study some simple properties of this rather non-trivial system.

(a) By inspecting the form of Eq. (1), explain qualitatively what will happen to $W(t)$ in the limit of large t for (i) $\alpha > 0$ and $|\alpha| \gg D$, and (ii) $\alpha < 0$ and $|\alpha| \gg D$.

(b) By directly integrating Eq. (1), find $W(t)$ in term of η. (Take the initial condition to be $W(t=0) = W_0$.)

(c) Compute $\langle W(t) \rangle$ by averaging the result of part **(b)** over η. Show that $\langle W(t) \rangle$ diverges (for large t) if α exceeds a critical value α_1. For $\alpha = \alpha_1 - \delta$, find how $\langle W(t \to \infty) \rangle$ diverges as $\delta \to 0^+$.
`Hint: It will be convenient to show and use the result`

$$\langle e^{\int d\tau y(\tau)\eta(\tau)} \rangle = e^{D \int d\tau y^2(\tau)}.$$

(d) Repeat part **(c)** for $\langle W^2(t) \rangle$. Compare and comment on your results for the two cases. What is the critical value α_2 for this case ?
`[Hint: There are some messy algebra to do if you want to make a complete calculation. However, you may focus on the leading singularity which will yield the critical value` α_2`.]`

To find the stationary distribution of W, perform a transformation: $W(t) = e^{h(t)}$.

(e) Write down the Langevin equation satisfied by $h(t)$. What is the corresponding Fokker-Planck equation ? Plot the "potential" $U(h)$ the dynamics of h is subject to. Show that there is no stationary solution (i.e., $h(t \to \infty)$ runs away) if α exceeds a critical value α_0.

(f) Find the stationary distribution $P_{\rm st}(h)$ for $\alpha < \alpha_0$. Plot the form of $P_{\rm st}(h)$. Compute $\langle h \rangle$ and $\langle h^2 \rangle$ for $\alpha = \alpha_0 - \delta$ and $\delta \to 0^+$. Comment on the differences between moments of W with moments of h.
`[Hint: in calculating movements of` $P_{\rm st}(h)$`, sketch the form of the distribution in the limit` $\delta \to 0^+$`. Convince yourself that the detail form of` $P_{\rm st}(h)$ `at` h `close to 0 does not affect the result to leading order in` δ`. Thus, you may replace` $P_{\rm st}(h)$ `by a form that is easy to integrate over.]`

(g) Use the defining relation between W and h to find the stationary distribution for W. Note that the distribution for W has *no scale* ! Use it to compute all integer moments $\langle W^m(t \to \infty)\rangle$.

(h) Comment on how the above results will be changed if the dynamics is of the Ito-kind.

`[In the context of investment return, one learns that even if the expected log-return` α `is negative (e.g., slot machine), there will still be a fraction of the population whose wealth becomes very large. Eq. (1) has been proposed as the simplest statistical model producing the broad wealth distribution observed in the real world.]`

2. **Driven dynamics in random potential** — In class, we formulated the problem of a particle driven by a force f in a one-dimensional random potential $U(x)$. You are asked here to derive and sketch the velocity-force characteristics for various forms of the random potential, characterized by the correlator

$$\widetilde{C}(x - x') \equiv \frac{1}{2}\langle [U(x) - U(x')]^2 \rangle = \Delta \cdot |x - x'|^\sigma .$$

Pay special attention to the small-f behavior.

(a) $\sigma > 1$.

(b) $1 > \sigma > 0$. You can perform the resulting integral by the "saddle-point" method for small f. Show that the linear mobility (i.e., $dv/df|_{f\to 0}$ vanishes but yet the particle moves forward for any finite f. `[This type of dynamics is known as ''creep'' and is a useful model of glassy relaxation.]`

(c) $\widetilde{C}(x - x') = \Delta \ln |x - x'|$. Does linear mobility exist in this case?

(d) $\sigma < 0$.