Phys 210B — Winter 2024

Problem Set #2

Due date: Wednesday Feb 14

1. **Levy-flights** — Consider a random walk of N steps in one-dimension. Suppose that the steps are completely uncorrelated, equally likely to be in either direction, and that the distribution of step size a is given by

$$\rho(a) \propto \frac{1}{1+|a|^\alpha}$$

where $\alpha > 0$ is a constant. We want to know the distribution $P(R,N)$ where $R = \sum_{n=1}^{N} a_n$.

(a) Write down the form of $P(R,N)$ expected from the Central Limit Theorem. Express parameters of P in terms of integrals of ρ. For what range of α might there be a problem? why?

(b) Compute the distribution $P(R,N)$ directly from $\rho(a)$ for $\alpha = 2$, and find its explicit form for all R's. How far does a walker typically go after N steps? Explain in words how this is possible given the form of ρ.

2. **Nonlinear Langevin equation** — In this problem, you will solve perturbatively a nonlinear stochastic equation

$$\frac{\partial h}{\partial t} = -\alpha h + \lambda h^2 + \eta(t),$$

where $\eta(t)$ is a Gaussian-distributed Langevin noise, with $\langle \eta \rangle = 0$ and

$$\langle \eta(t)\eta(t') \rangle = 2D\delta(t-t').$$

You can assume that the initial time t_0 is far away, i.e., $t_0 \to -\infty$.

(a) Rewrite the equation of motion in terms of the Fourier transforms

$$\hat{h}(\omega) = \int dt\, h(t)e^{i\omega t} \qquad \text{and} \qquad \hat{\eta}(\omega) = \int dt\, \eta(t)e^{i\omega t}.$$

Work out the noise correlator $\langle \hat{\eta}(\omega)\hat{\eta}(\omega') \rangle$.

For the temporal correlation function $\langle h(t)h(t') \rangle \equiv C(t-t')$, show that the Fourier transforms are related by

$$\langle \hat{h}(\omega)\hat{h}(\omega') \rangle = \hat{C}(\omega)\, 2\pi\, \delta(\omega+\omega')$$

where $\hat{C}(\omega)$ is the Fourier Transform of $C(t)$.

(b) For $\lambda = 0$, the equation can be written as

$$\hat{h}_0(\omega) = \hat{G}_0(\omega)\hat{\eta}(\omega).$$

Find the form of $\hat{G}_0(\omega)$. Write the correlation function $\hat{C}_0(\omega)$ in term of $\hat{G}_0$.

(c) Rewrite the equation of motion for $\hat{h}(\omega)$ in (a) using the form of $\hat{G}_0$. Inserting

$$\hat{h}_2(\omega) = \hat{h}_0(\omega) + \lambda\, \delta\hat{h}_1(\omega) + \lambda^2\, \delta\hat{h}_2(\omega)$$

into the equation of motion, find $\hat{h}_2$ as a sum involving integrals of $\hat{G}_0$ and $\hat{\eta}$ to order λ^2.

(d) Find the the first moment of h, $\langle h \rangle$, to $O(\lambda^2)$. Express the result in terms of integral of $\hat{C}_0$ and find the value in terms of the parameters λ, D, α. What is the meaning of this result? [It will be useful to express things back in real time to see this.]

The response function,

$$\hat{G}(\omega) \equiv \left\langle \frac{\delta \hat{h}(\omega)}{\delta \hat{\epsilon}(\omega)} \right\rangle_{\hat{\epsilon}\to 0},$$

is obtained by adding a perturbation $\epsilon(t)$ to the equation of motion for h, or equivalent, by making the change $\hat{\eta}(\omega) \to \hat{\eta}(\omega) + \hat{\epsilon}(\omega)$ in the integral equation for $\hat{h}(\omega)$.

(e) Show that $\hat{G}(\omega) = \hat{G}_0(\omega)$ if $\lambda = 0$.

(f) Show to $O(\lambda^2)$ that $\hat{G}_2(\omega) \equiv \left\langle \delta\hat{h}_2(\omega)/\delta\hat{\epsilon}(\omega) \right\rangle_{\hat{\epsilon}\to 0}$ can be expressed as

$$\hat{G}_2(\omega) = \hat{G}_0(\omega) + \hat{G}_0(\omega) \cdot \hat{\Pi}(\omega) \cdot \hat{G}_0(\omega),$$

or equivalently, $\hat{G}_2^{-1}(\omega) = \hat{G}_0^{-1}(\omega) - \hat{\Pi}(\omega)$ for small λ. Find the leading ω dependence of the real and imaginary components of $\hat{\Pi}(\omega)$ in the limit of small ω. Express your result as $\hat{G}_2^{-1}(\omega) = \gamma \cdot (\widetilde{\alpha} - i\omega)$, and give the forms of the "renormalized" parameters γ and $\widetilde{\alpha}$.

The correlation function is defined as $C(t - t') \equiv \langle [h(t) - \langle h \rangle] \cdot [h(t') - \langle h \rangle] \rangle$.

(g) Show that to $O(\lambda^2)$, the Fourier Transform of the correlation function has the form

$$\hat{C}_2(\omega) = \hat{C}_0(\omega) \cdot [1 + \lambda^2 I(\omega)],$$

where $I(\omega)$ contains several terms each involving $\hat{C}_0$, $\hat{G}_0$, and/or their integrals.

You should know that the fourth moment of a Gaussian distributed random varaible η is given in terms of the second moment as:

$$\langle \eta_1\eta_2\eta_3\eta_4 \rangle = \langle \eta_1\eta_2 \rangle\langle \eta_3\eta_4 \rangle + \langle \eta_1\eta_3 \rangle\langle \eta_2\eta_4 \rangle + \langle \eta_1\eta_4 \rangle\langle \eta_2\eta_3 \rangle.$$

(h) Without evaluating the integrals, show that the above form of $\hat{C}_2(\omega)$ can be expressed equivalently as

$$\hat{C}_2(\omega) = 2\tilde{D}(\omega)|\hat{G}_2(\omega)|^2,$$

to $O(\lambda^2)$. Compare to your answer in part (b) and compute the renormalized noise amplitude $\widetilde{D} \equiv D(\omega \to 0)$.

3. **Fokker-Planck equation with continuous-time dynamics** — Consider the following Langevin equation

$$\frac{dr}{dt} = f(r) + \eta(t) \tag{1}$$

where $\eta(t)$ is Gaussian distributed, with $\langle \eta \rangle = 0$ and

$$\langle \eta(t)\eta(t') \rangle = \frac{D}{\tau_0} e^{-|t-t'|/\tau_0}. \tag{2}$$

(a) For a multi-variable Gaussian distribution

$$P_0[\eta] = \mathcal{N}^{-1} e^{-\frac{1}{2}\int d\tau d\tau' K(\tau,\tau')\eta(\tau)\eta(\tau')}$$

where $\mathcal{N}$ is a normalization constant and the kernel $K(\tau, \tau')$ is symmetric and invertible, show that

$$\langle \eta(t) F[\eta] \rangle = \int d\tau K^{-1}(t,\tau) \left\langle \frac{\delta F}{\delta \eta(\tau)} \right\rangle \tag{3}$$

for an arbitrary *functional* $F[\eta]$. Use $F[\eta] = \eta(t')$ in Eq. (3) to find K^{-1} in term of $\langle \eta\eta \rangle$ given in (2).

(b) Derive the Fokker-Planck equation which describes the time evolution of the probability density $P(x,t) = \langle \delta(x - r(t)) \rangle$ by taking the time derivative of $P(x,t)$ and then directly performing statistical average of the resulting expression in the limit $\tau_0 \to 0$.

(c) Repeat the above for multiplicative noise, i.e., for Langevin equation of the form

$$\frac{dr}{dt} = f(r) + g(r) \cdot \eta(t), \tag{4}$$

and the same noise as that characterized by Eq. (2). Show that for $g(r) \neq$ constant, the effective drift term in the resulting Fokker-Planck equation is no longer the force f.