**Phys 210B — Winter 2024**
**Problem Set #1: Kinetic Theory & the Boltzmann Equation**
**Due date: Wednesday January 24**

1. **Evolution of the density function in phase space** — A thermalized classical gas is confined to a one-dimensional trap. Suppose the particles of this gas do not collide so that the gas can be described by an ensemble of "1d boxes" each containing a *single* thermalized gas particle. Let the initial density function of this gas be $\rho(r, p, t = 0) = \delta(r) f(p)$, where $f(p) \propto \exp(-p^2/2mk_BT)$.

   **(a)** Starting from Liouville's equation, derive $\rho(r, p, t)$ and sketch it in the $(r, p)$ plane for several $t$.

   **(b)** Derive the expressions for the average $\langle r^2 \rangle$ and $\langle p^2 \rangle$ for $t > 0$.

   **(c)** Suppose that hard "walls" are placed at $r = \pm R$. Sketch $\rho(r, p, t)$ again for several $t$ and describe its behavior for $t \gg \tau$, where $\tau$ is an appropriately large relaxation time.

   **(d)** A "coarse-grained" density $\tilde{\rho}$ is obtained by ignoring variations of $\rho$ below some small resolution in the $(r, p)$ plane; for example, by averaging $\rho$ over cells of the resolution area. Find $\tilde{\rho}(r, p)$ for the situation in part (c), and show that it is stationary.

2. **Maximum entropy** — Given the density function $\rho(\Gamma, t)$ defined in class, the associated entropy of the distribution is

$$S(t) = -\int d\Gamma\, \rho(\Gamma, t)\, \ln \rho(\Gamma, t).$$

   **(a)** Show that if $\rho(\Gamma, t)$ satisfies Liouville's equation for a Hamiltonian $\mathcal{H}$, then $dS/dt = 0$.

   **(b)** Using the method of Lagrange multipliers, find the function $\rho_{\rm max}(\Gamma)$ that maximizes the functional $S[\rho]$, subject to the constraint of fixed average energy, $\langle \mathcal{H} \rangle = \int d\Gamma\, \rho\, \mathcal{H} = E$.

   **(c)** Show that the solution to part (b) is stationary, that is $\partial \rho_{\rm max}/\partial t = 0$.

   This approach of finding a stationary distribution is called the "maximal entropy" approach.

3. **Two-component plasma** — Consider a *neutral* mixture of $N$ ions of charge $+e$ and mass $m_+$, and $N$ electrons of charge $-e$ and mass $m_-$, in a volume $V$.

   **(a)** Show that the Vlasov equations for this two-component system are

$$\left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m_+} \cdot \vec{\nabla}_{\vec{r}} - e\, \vec{\nabla}_{\vec{r}}\, \Phi_{\rm eff} \cdot \vec{\nabla}_{\vec{p}}\right] f_+(\vec{p}, \vec{r}, t) = 0 \qquad (1)$$

$$\left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m_-} \cdot \vec{\nabla}_{\vec{r}} + e\, \vec{\nabla}_{\vec{r}}\, \Phi_{\rm eff} \cdot \vec{\nabla}_{\vec{p}}\right] f_-(\vec{p}, \vec{r}, t) = 0 \qquad (2)$$

   where the effective Coulomb potential is given by

$$\Phi_{\rm eff}(\vec{r}, t) = \Phi_{\rm ext}(\vec{r}) + e \int d^3\vec{r}\,' d^3\vec{p}\,' C(\vec{r} - \vec{r}\,') \left[f_+(\vec{r}\,', \vec{p}\,', t) - f_-(\vec{r}\,', \vec{p}\,', t)\right].$$

   Here, $\Phi_{\rm ext}$ is the potential set up by the external charges, and the Coulomb potential $C(\vec{r})$ satisfies the differential equation $\nabla^2 C = -4\pi \delta^3(\vec{r})$.

**(b)** Assume that the one-particle densities have the sationary forms $f_\pm = g_\pm(\vec{p}) \cdot n_\pm(\vec{r})$. Show that the effective potential satisfies the equation

$$\nabla^2 \Phi_{\rm eff} = -4\pi\rho_{\rm ext}(\vec{r}) - 4\pi e \left[n_+(\vec{r}) - n_-(\vec{r})\right],$$

where $\rho_{\rm ext}$ is the external charge density.

**(c)** The *Poisson-Boltzamann equation* is obtained upon further assuming that the densities relax to the equilibrium Boltzmann weights, i.e., $n_\pm(\vec{r}) = n_0 \exp[\mp\beta e \Phi_{\rm eff}(\vec{r})]$, where $n_0 \equiv N/V$. It is not possible to solve the Poisson-Boltzmann equation generally due to its nonlinear form. Linearize the exponentials to obtain the simpler *Debye equation*,

$$\nabla^2 \Phi_{\rm eff} = 4\pi\rho_{\rm ext}(\vec{r}) + \Phi_{\rm eff}(\vec{r})/\lambda^2.$$

Give the expression for the *Debye screening length* $\lambda$.

**(d)** Show that the Debye equation has the general solution

$$\Phi_{\rm eff}(\vec{r}) = \int d^3\vec{r}'\; G(\vec{r}-\vec{r}')\; \rho_{\rm ext}(\vec{r}'),$$

where $G(\vec{r}) = \exp(-|\vec{r}|/\lambda)/|\vec{r}|$ is the *screened Coulomb potential.*

**(e)** Give condition for the self-consistency of the Vlasov approximation in terms of the particle spacing $\ell \propto n_0^{-1/3}$ and other constants such as $k_B T$. Explain your result in words.

4. **Boltzmann's H-theorem**

**(a)** Extend the proof of Boltzmann's H-theorem given in class to the case with an external potential $U(\vec{r})$.

**(b)** Prove the H-theorem for the quantum Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \frac{\vec{p}_1}{m}\cdot\vec{\nabla}_{r_1} + \vec{F}\cdot\vec{\nabla}_{p_1}\right) f(\vec{r}_1,\vec{p}_1,t)$$
$$= \int d^3p_2\, d^3p_1'\, d^3p_2'\; T(\vec{p}_1,\vec{p}_2|\vec{p}_1',\vec{p}_2')\cdot$$
$$\cdot\Big[f(1')f(2')(1\pm h^3 f(1))(1\pm h^3 f(2)) - f(1)f(2)(1\pm h^3 f(1'))(1\pm h^3 f(2'))\Big],$$

where "+" refers to bosons, "-" refers to fermions, and $h$ is the Planck constant, by constructing an appropriate functional $H[f]$.

`[Hint: try` $H(t) = \int d^3r d^3p\, g[f(r,p,t)]$`, carry on the proof done in a class for as long as possible, until at a point where a form of` $g[f]$ `has to be taken to facilitate the proof.]`

**(c)** Derive the equilibrium distribution functions for the quantum kinetic gases (fermions and bosons) subject to an external potential $U(\vec{r})$.