Model: 1-dim Random Walk

\[
\frac{dr}{dt} = -v + \gamma(t), \quad v \geq 0
\]

\[
\langle \gamma(t) \gamma(t') \rangle = 20D(t-t')
\]

\[
P(r,t|r',t') = \frac{1}{\sqrt{4\pi D(t-t')}} \exp \left( -\frac{(r-r'+v(t-t'))^2}{4D(t-t')} \right)
\]

\[
\rightarrow \text{want } L(c) = \text{prob that RW exceeds } c \text{ at some time}
\]

(a) no drift \( v = 0 \)

let \( Q_c(r,t) \) be prob fr particle to be at \( (r,t) \)

and never have been beyond \( c \)

then \( L(c) = 1 - \int_{-\infty}^{c} dr \ Q_c(r,t \rightarrow \infty) \)

\( Q_c(r,t) \) : soln of diff eqn

with \( \text{as } Q(r=c) = 0 \)

Can be solved using "image charge"

Every path from \( (0,0) \) to \( (r,t) \) which

has gone beyond \( c \) is canceled by the path from \( (2c,0) \) to \( (r,t) \)

\[
Q_c(r,t) = P(r,t|0,0) - P(r,t|2c,0)
\]

\[
= \frac{1}{\sqrt{4\pi D t}} \exp \left( -\frac{r^2}{4Dt} \right) - \frac{1}{\sqrt{4\pi D t}} \exp \left( -\frac{(r-2c)^2}{4Dt} \right)
\]
\[
\int_{-\infty}^{c} Q_c(r,t)dr = \frac{1}{\sqrt{4\pi Dt}} \left[ \int_{-\infty}^{c} e^{-\frac{(r-\bar{c})^2}{4Dt}} dr - \int_{-\infty}^{c} e^{-\frac{(r-c)^2}{4Dt}} dr \right] \\
= \frac{1}{\sqrt{4\pi Dt}} \int_{-c}^{c} e^{-\frac{y^2}{4Dt}} dy = \sqrt{\frac{2}{\pi Dt}} \rightarrow 0
\]

\[\Rightarrow Q(c) \rightarrow 1\]

with drift \(v=0\), any finite boundary \(c\) is passed with probability 1.

(b) with drift \(v>0\) (downward)

\[\Rightarrow \text{use "first passage time" method}\]

let \(g_c(t') = \text{prob to hit } c \text{ for the first time at time } t'\)

then \(Q(c) = \int_{0}^{\infty} g_c(t')dt'.\)

to find \(g_c(t)\), consider \(P(r,t;\infty)\) with \(r>c\)

the path has to hit \(c\) at some point \(t'<t\) for first time

\[\Rightarrow P(r,t|1,0,0) = \int_{0}^{t'} g_c(t') P(r,t'|c,t') dt'\]

Can solve for \(g_c(t)\) in Laplace space

for \(P(r,s) = \int_{0}^{\infty} e^{-ts} P(r,t) dt\) and \(g_c(s) = \int_{0}^{\infty} e^{-ts} g_c(t) dt\)

we have \(\hat{P}(r,s) = \hat{g}_c(s) \cdot \hat{P}(r-c,s)\)
Find \( \hat{P}(r,s) \) via its Fourier Transform \( \hat{F}(r,s) \):

\[
\hat{P}(r,s) = \int dk \frac{e^{-ikr}}{Dk^2 + i kv + s}
\]

where \( \hat{F}(k,s) = \int dt e^{ts} \int dr e^{ikr} \frac{1}{\sqrt{4\pi t}} e^{-\frac{r^2}{4at}} \).

Transform \( \beta \)

\[
\hat{F}(k,s) = \int dt e^{ts} \int dr \frac{e^{-\frac{r^2}{4at}}}{\sqrt{4\pi at}} e^{ikr} e^{-ikvt} e^{-Dk^2 t}
\]

\[
= \frac{1}{Dk^2 + i kv + s}
\]

\[
\hat{P}(r,s) = \int dk \frac{e^{-ikr}}{Dk^2 + i kv + s}
\]

\[
= i \text{ Res} \left( \frac{e^{-ikr}}{Dk^2 + i kv + s} \right)
\]

Poles of \( \hat{F}(k,s) \): \( Dk^2 + i kv + s = 0 \)

\[
k_0 = -i\left( \frac{u \pm \sqrt{v^2-4Ds}}{2D} \right) = -i \left( \frac{u \pm \sqrt{\frac{v^2}{2D} + \frac{4s}{D}}}{} \right)
\]

\[
\hat{P}(r,s) = i \text{ Res} \left( \frac{e^{-ikr}}{D(k-i\kappa)(k+i\kappa)} \right) = \frac{i e^{-k\kappa}}{-D x(k+i\kappa)}
\]

\[
= e^{-\left( \frac{kr}{2D} + \frac{\sqrt{(kr)^2 + \kappa^2}}{2D} \right)}
\]

\[
D \left( \frac{kr}{2D} \right)
\]

\[
\hat{g}_c(s) = \frac{\hat{P}(r,s)}{\hat{P}(r-c,s)} = e^{-\left( \frac{kr}{2D} + \sqrt{\frac{(kr)^2 + \kappa^2}{2D}} \right)} c
\]

\[
\hat{g}_c(s) = e^{-\left( \frac{kr}{2D} + \sqrt{\frac{(kr)^2 + \kappa^2}{2D}} \right)} c
\]

\[
L(c) = \int_0^\infty dt \hat{g}_c(t) = \hat{g}_c(s=0) = e^{-\frac{kr}{2D}} c
\]
(c) Extremal ensemble
Consider discrete set of variables
e.g., $\sigma_n \in \{ s_1, s_2, \ldots, s_k \}$
with probabs. $p_1, p_2, \ldots, p_k$ respectively
and $\sum_{i=1}^k p_i = 1$
Suppose $S = \sum_{i=1}^k p_i s_i = -|\nu| < 0$
but $X_N = \sum_{n=1}^N \sigma_n = c > 0$ for $N \gg 1$

c
\[ \text{typical} \]
\[ \text{rare} \]

What is the likely composition of the rare event $X_N > c$?
Quantify: $X_N = n_1 s_1 + n_2 s_2 + \ldots + n_k s_k$
where $n_i = \# \text{time } s_i \text{ appeared in } X_N > c$.

$p_i = \frac{n_i}{N} = \text{composition of rare event}$
(Extremal Ensemble)
(c.f. microcanonical ensemble)

Probability of occurrence:

$P(n_1, n_2, \ldots n_k) = \frac{(\sum n_i)!}{n_1! n_2! \ldots n_k!} \frac{k!}{i=1} p_i^{n_i}$
Use Stirling approx: \[ \ln \Gamma = \Gamma \ln \Gamma - \Gamma \]
\[ \ln \Omega = (\sum_i n_i) \ln (\sum_i n_i) - \sum_i n_i \]
\[ \quad - \sum_i (n_i \ln n_i - n_i) + \sum_i n_i \ln \pi_i \]

Constraint: \[ \sum_i n_i \pi_i = c \]

Most likely dist \( P^* \) → maximize \( P \) subject to constraint

→ use Lagrange multiplier: \[ L = \ln P + \lambda \left( \sum_i n_i - c \right) \]

\[ \frac{\partial L}{\partial n_i} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial n_i} \ln P + \lambda \frac{\partial}{\partial n_i} n_i = 0 \]

\[ \ln (\sum_i n_i^*) + 1 - \ln n_i^* - 1 + \ln \tilde{p}_i + \lambda n_i = 0 \]

\[ \ln \tilde{p}_j + \lambda n_j = \ln \frac{n_j^*}{\sum_i n_i^*} \]

\[ \Rightarrow \quad \frac{n_j^*}{\sum_i n_i^*} = \tilde{q}_j = \frac{p_j e^{\lambda n_j}}{\sum_j p_j e^{\lambda n_j}} \]

\[ \text{det} \sum_j \tilde{q}_j = 1 \Rightarrow \sum_j p_j e^{\lambda n_j} = 1 \]

HW 4: show \[ S^* = \sum_j \tilde{q}_j \sigma_j > 0 \]

\[ L(c) = e^{-\lambda c} \]

For Gaussian dist: \[ \lambda = \frac{\nu}{\sigma^2} \]