Last time: driven transport in random potential.

\[ \frac{dx}{dt} = \Gamma F(x) + \gamma(t) \]

\[ F(x) = -\frac{dU}{dx} + f \]

Random potential:

Gaussian w/ 2nd moment

\[ \langle U(x) U(x') \rangle = C(x-x') \]

\[ \frac{1}{2} \langle [U(x) - U(x')]^2 \rangle = \hat{C}(x-x') = C(0) - C(x) \]

\[ \frac{dx}{dt} = \Gamma F(x) + \gamma(t) \]

\[ F(x) = -\frac{dU}{dx} + f \]

\[ \rightarrow \text{relevant response: } \bar{U} = \int_0^L dxf(x) \]

Since \( J = P_s(x) \cdot V(x) \rightarrow \bar{V} = \frac{P_s}{J} = \frac{1}{J L} \)

\[ \bar{V} = \sqrt{J} = J : L \]  \[ \text{want } \bar{V}(f) \propto J(f) \]

Find:

\[ \bar{U} = (J L)^{-1} = \frac{1}{L} \int_0^{L \beta} dE e^{-\frac{\beta E}{2}} \frac{1}{L} \int_0^L dx e^{-\beta [U(x+z)-U(x)]} \]

\[ \bar{U} = \frac{L}{\beta} \int_0^{\beta z} e^{-\beta z} + \beta^2 \hat{C}(z) \]

for \( \hat{C}(z) = \Delta |z| \text{ (r.w.)} \)

\[ \hat{U} = \sum \begin{cases} \Gamma (f - f_c) & f > f_c = \beta \Delta \\ 0 & f < f_c \end{cases} \]

Statistical

\[ \langle \bar{U} \rangle = \frac{1}{\beta} \int_0^{\beta z} e^{-\beta z} \right] \]

\[ = \frac{1}{\beta} \int_0^{\beta z} e^{-\beta z} \beta^2 \left[ \int \frac{dU}{\mathbb{U}} e^{-\beta [U(x+z)-U(x)]} \right] \]

\[ = \frac{1}{\beta} \int_0^{\beta z} e^{-\beta z} \beta^2 \left[ \int \frac{dU}{\mathbb{U}} e^{-\beta [U(x+z)-U(x)]} \right] \]

\[ = \beta^2 \hat{C}(z) \]

\[ \hat{U} = \sum \begin{cases} \Gamma (f - f_c) & f > f_c = \beta \Delta \\ 0 & f < f_c \end{cases} \]
B. Time-dependent soln

1. General soln (one variable) via transformation

\[ \frac{\partial P}{\partial t} = -\frac{2}{\partial x} \left[ f(x) P(x,t) \right] + \frac{\partial^2}{\partial x^2} \left[ g(x) P(x,t) \right] \]

given \( P(x,t=0) = P_0(x) \)

how does \( P(x,t) \) approach \( P_{st}(x) \)?

i) Convert to additive noise

→ find a transformation \( x \rightarrow z(x) \) to remove \( f(x) \)

\[ P(x,t) \rightarrow Q(z,t) \]

\[ P(x,t) \, dx = Q(z,t) \, dz \]

→ \( Q(z,t) = P(z) \cdot \beta(z) \) where \( \beta = \frac{dx}{dz} \).

\[ \frac{\partial Q}{\partial t} = \beta(x) \frac{\partial P}{\partial t} = -\beta \frac{2}{\partial x} \left[ f(x) P \right] + \beta \frac{\partial^2}{\partial x^2} \left[ g(x) P \right] \]

\[ \frac{\partial}{\partial x} \left( gP \right) = \frac{9}{\beta^2} \frac{\partial}{\partial z} \left[ g \frac{\partial}{\partial z} \right] = \frac{9}{\beta^2} \frac{\partial}{\partial x} + \frac{1}{\beta^2} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial z} \right) \cdot Q \]

\[ \frac{\partial Q}{\partial t} = \frac{9}{\beta^2} \left[ -\frac{\partial}{\partial x} \left( fQ \right) + \frac{\partial^2}{\partial x^2} Q + \frac{9}{\beta^2} \frac{\partial}{\partial x} \left( fQ \right) \right] \]

→ \( \frac{\partial}{\partial x} Q = -\frac{2}{\partial x} \left( \frac{\partial}{\partial x} fQ \right) + \frac{\partial^2}{\partial x^2} Q \)

with \( \tilde{f} = \frac{f}{\beta} - \frac{1}{\beta} \frac{\partial}{\partial x} P \), \( \beta = g^{1/2} \)
ii) Solve for FP eqn with additive noise

\[ \frac{3}{2} \dot{\Omega} = -\frac{3}{2} [\hat{\mathcal{F}}(\Omega)] + \frac{3^2}{2^2} \Omega \]

To remove \( \frac{3}{2} [\hat{\mathcal{F}}(\Omega)] \) term, try \( \Omega(z,t) = \pi(z) \psi(z,t) \)

\[ \dot{\Omega} = \pi' \psi + \pi \psi' \]
\[ \ddot{\Omega} = \pi'' \psi + 2 \pi' \psi' + \pi \psi'' \]
\[ \dot{\Omega} = \ddot{\Omega} - \hat{\mathcal{F}} \Omega - \hat{\mathcal{F}} \Omega' \]
\[ \pi' \psi = \pi'' \psi + 2 \pi' \psi' + \pi \psi''' - \pi'' \psi - \hat{\mathcal{F}} \left( \pi' \psi + \pi \psi' \right) \]

Coeff of \( \psi \) term:

\[ \hat{f}' \pi + \hat{f} \pi' = (2 \pi') \pi + 2 \pi' \pi = 2 \pi' - \frac{2 \pi'}{2 \pi} \pi + \frac{2 \pi'}{2 \pi} \pi = 2 \pi'' \]

\[ \Rightarrow \dot{\psi} = \psi'' - \frac{\pi'}{\pi} \psi \]

Imaginary time Schrödinger eqn

with "potential" \( V(z) = \frac{\pi''}{\pi} = \frac{1}{2} \hat{\mathcal{F}}' + \frac{1}{2} \pi' \frac{\pi'}{\pi} = \frac{1}{2} \hat{\mathcal{F}}' + (\pi' \pi) \]

\[ \Rightarrow \text{dynamics obtained from solving the eigenvalue prob.} \]
Several simple systems (exactly soluble)

1. Harmonic spring: \( \frac{\partial P}{\partial t} = -2\lambda x (\partial \partial_x P) + D \frac{\partial^2 P}{\partial x^2} \).

   \[ f(x) = -kx \quad P(x, t=0) = S(x-x_0) \]

   \[ \frac{dx}{dt} = -ky + \gamma(t) \quad P_{xt}(x) = \frac{1}{\sqrt{2\pi} \sigma(t)} e^{-\frac{x^2}{2\sigma^2(t)}} \]

   Full soln: \( P(x,t) = \frac{1}{\sqrt{2\pi} \sigma(t)} e^{-\frac{x^2}{2\sigma^2(t)}} \)

   With \( \sigma^2(t) = \frac{\gamma}{\gamma} (1-e^{-\gamma t}) \)

2. Sedimentation:

   \[ \frac{dh}{dt} = -\rho g + \gamma(t) \]

   \[ \frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial z^2} + \nu \frac{\partial \partial_z P} \]

   \( P(z, t=0) = S(z-z_0) \)

   \( \rightarrow P_{zt}(z) \propto e^{-\frac{\rho z}{h_0}} \)

   B.C. \( P(z=\infty) = 0 \quad J(z=0) = 0 \)

2. Rare events

   Consider queuing request

   \[ r_i = \# \text{ request/time (random)} \]

   \( \nu = \text{processing rate} \)

   \( S_i = r_i - \nu \quad \# \text{ request stored in memory/time} \)

   Memory size:

   \[ M_{i+1} = \max \{ M_i + S_i, 0 \} \]

   What is probab that \( M \) reaches \( c \)?
Gambling

\( \eta_i = \text{amt paid out/game}, \langle \eta \rangle = 0 \)
\( V = \text{fixed amt received/game}, V > 0 \)
\( X_i = \text{total amt available} \)

→ start with C, what is prob that house will break?

Let fluctuation be characterized by Gaussian random process with \( \text{Var}(\eta) = D \).
- Typical fluctuation after \( N \) tries: \( \sqrt{DN} \)
- Deterministic gain: \( VN \)

→ typical bad case:

\[ X = VN - \sqrt{DN} \]

\( x^* \) from \( \frac{dx}{dN} = 0 = V - \sqrt{DN} \)

\[ N^* \sim \frac{D}{V^2}, \quad X^* \sim \frac{D}{V} \]

Set \( C \Rightarrow x^* = D/V \)

Then chance \( X_i \) reaches \( C \) is very small ("rare events")

but, given long enough time, disaster will happen!