

Physics 210A — Spring 2024
Problem Set #4 Solutions

1. Liquid-gas transition

(a) The Van der Waal's equation of state is

$$\left(P + \frac{a}{v^2}\right)(v - v_0) = k_B T. \quad (1)$$

To find the critical point we set the derivatives $\frac{\partial P}{\partial v}$ and $\frac{\partial^2 P}{\partial v^2}$ equal to zero, which gives the conditions

$$\begin{aligned} \frac{k_B T_c}{(v_c - v_0)^2} &= \frac{2a}{v_c^3}, \\ \frac{2k_B T_c}{(v_c - v_0)^3} &= \frac{6a}{v_c^4}. \end{aligned} \quad (2)$$

Solving this system of equations, we find

$$v_c = 3v_0, \quad T_c = \frac{8a}{27k_B v_0^2}, \quad P_c = \frac{a}{27v_0^2}, \quad (3)$$

and

$$\frac{P_c v_c}{k_B T_c} = \frac{3}{8}. \quad (4)$$

In terms of the dimensionless variables $\tilde{P} = P/P_c$, $\tilde{v} = v/v_c$, $\tilde{T} = T/T_c$, we have

$$\left(\tilde{P} + \frac{3}{\tilde{v}^2}\right)(3\tilde{v} - 1) = 8\tilde{T}. \quad (5)$$

(b) The variational free energy is (up to an arbitrary constant)

$$\Psi = - \int dv' \left[\tilde{P}(v') - \tilde{P} \right] = -\frac{8}{3}\tilde{T} \ln\left(\tilde{v} - \frac{1}{3}\right) - 3/\tilde{v} + \tilde{P}\tilde{v}. \quad (6)$$

In Fig. 1, we plot Ψ versus \tilde{v} for $(\tilde{P}, \tilde{T}) = (0.5, 0.5)$ and $(\tilde{P}, \tilde{T}) = (1, 1)$. At the critical point Ψ becomes flat since there is a continuous phase transition.

(c) We have $\tilde{T} = 1 + t$, $\tilde{P} = 1 + p$, and $\tilde{v} = 1 + \nu$, so

$$\begin{aligned} \Psi &= -\frac{8}{3}(1+t) \ln\left(1 + \frac{3}{2}\nu\right) - 3(1+\nu)^{-1} + (1+p)(1+\nu). \\ &= (p - 4t)\nu + 3t\nu^2 - 3t\nu^3 + \frac{1}{8}(3 + 27t)\nu^4 + \dots \end{aligned} \quad (7)$$

The corresponding equation of state is

$$p - 4t + 6t\nu - 9t\nu^2 + \frac{1}{2}(3 + 27t)\nu^3 = 0. \quad (8)$$

The coexistence curve is determined from $\Psi(\nu_G) = \Psi(\nu_L)$, where Ψ has minima at ν_G and ν_L corresponding to the gas and liquid phases, respectively. To leading order in t , it is given by setting the linear term in Ψ equal to zero, which gives

$$p = 4t. \quad (9)$$

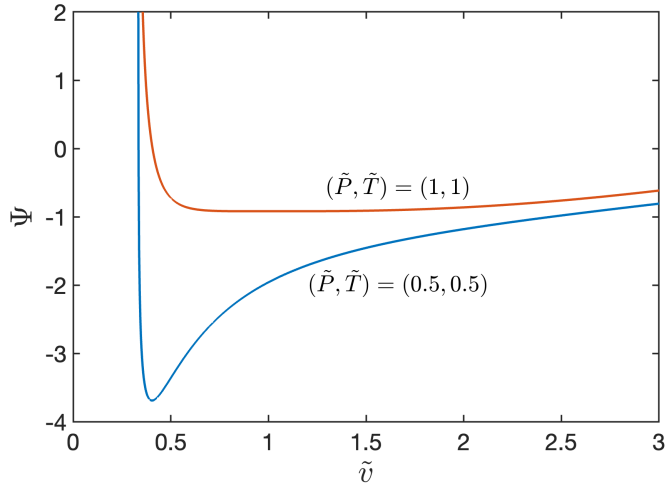


Figure 1: Free energy Ψ versus \tilde{v} for two different values of (\tilde{P}, \tilde{T}) .

(d) Along the coexistence curve, the equation of state becomes

$$-6t\nu + 9t\nu^2 - \frac{3}{2}\nu^3 = 0. \quad (10)$$

To leading order in t , we neglect the quadratic term, so the two nonzero roots are $\nu_{\pm} = \pm 2\sqrt{-t}$. The difference in specific volume of the two phases is therefore

$$\Delta\nu = 4\sqrt{-t}, \quad (11)$$

so $\beta = \frac{1}{2}$. From the Clausius-Claperon equation, we find the latent heat

$$\Delta s = \frac{dp}{dt} \Delta\nu = 4\Delta\nu = 16\sqrt{-t}, \quad (12)$$

in units of $k_B T_c$.

(e) For $t = 0$, the equation of state is

$$-p - \frac{3}{2}\nu^3 = 0, \quad (13)$$

so

$$\nu \sim p^{1/3}, \quad (14)$$

and $\delta = 3$.

(f) Differentiating the equation of state with respect to p , we find to leading order in t

$$-1 - (6t - \frac{9}{2}\nu^2) \left(\frac{\partial\nu}{\partial p} \right)_t = 0. \quad (15)$$

For $t > 0$, we have $\nu = 0$, whereas for $t < 0$, we have $\nu = \pm 2\sqrt{-t}$. Therefore,

$$\kappa_T = \begin{cases} \frac{1}{6t} & t > 0 \\ \frac{1}{12|t|} & t < 0 \end{cases}, \quad (16)$$

and $\gamma = 1$. There is one more critical exponent, α , which is defined by $c_V \propto |t|^{-\alpha}$. It is left as an exercise to the reader to show that $\alpha = 0$! In Table 1, we compare our

Exponent	MFT	3D Ising (numerical)	CO ₂ (exp.)
α	0	0.125	$\lesssim 0.1$
β	1/2	0.313	0.35
γ	1	1.25	1.26
δ	3	5	4.2

Table 1: Comparison of mean-field theory (MFT) results for the critical exponents with numerical results for the 3D Ising model and experiments on the liquid-gas transition in CO₂. *Source*: Daniel Arovas's 210A lecture notes.

mean-field values of the critical exponents with the exact numerical values for the 3D Ising model (in the same universality class as the liquid-gas transition) and experimental values for the liquid-gas transition in CO₂. We see that the mean-field predictions are reasonably close to the correct values.

2. Antiferromagnetism

- (a) The square lattice is a bipartite lattice, which means it can be divided into two sublattices, which we call A and B, such that A has only B neighbors, and B has only A neighbors. At $T = 0$ and $H = 0$, the two ground state spin configurations are

$$s_i = \begin{cases} \pm 1 & i \in A \\ \mp 1 & i \in B \end{cases}. \quad (17)$$

The energy of these configurations is

$$E_0 = -J \sum_{\langle ij \rangle} = -2NJ. \quad (18)$$

At $T = 0$ and finite H , we compare this with the configuration where all the spins are aligned with the field, which has energy

$$E_1 = J \sum_{\langle ij \rangle} - H \sum_i = N(2J - H). \quad (19)$$

For $H > H_c = 4J$, we have $E_1 > E_0$, so the ground state has all the spins aligned.

- (b) To find the mean-field Hamiltonian, we decompose s_i for each sublattice into an average term $\bar{s}_{A,B}$ and a fluctuation δs_i :

$$s_i = \begin{cases} \bar{s}_A + \delta s_i, & i \in A \\ \bar{s}_B + \delta s_i, & i \in B \end{cases}. \quad (20)$$

The interaction term becomes

$$\begin{aligned} s_i s_j &= (\bar{s}_{A,B} + \delta s_i)(\bar{s}_{B,A} + \delta s_j) \\ &= \bar{s}_A \bar{s}_B + \bar{s}_{A,B} \delta s_j + \bar{s}_{B,A} \delta s_i + \delta s_i \delta s_j \\ &= -\bar{s}_A \bar{s}_B + \bar{s}_{A,B} s_j + \bar{s}_{B,A} s_i + \delta s_i \delta s_j. \end{aligned} \quad (21)$$

The mean-field approximation means we neglect the $\delta s_i \delta s_j$ term in Eq. (21). Hence, the mean-field Hamiltonian is

$$\mathcal{H}_{\text{MF}} = -2NJ\bar{s}_A\bar{s}_B + (4J\bar{s}_B - H) \sum_{i \in A} s_i + (4J\bar{s}_A - H) \sum_{i \in B} s_i. \quad (22)$$

The partition function is $Z = Z_1^{N/2}$ where

$$\begin{aligned} Z_1 &= \sum_{s_A=\pm 1} \sum_{s_B=\pm 1} = \exp \{ -\beta [-4J\bar{s}_A\bar{s}_B + (4J\bar{s}_B - H)s_A + (4J\bar{s}_A - H)s_B] \} \\ &= 4e^{4\beta J\bar{s}_A\bar{s}_B} \cosh[\beta(4J\bar{s}_A - H)] \cosh[\beta(4J\bar{s}_B - H)]. \end{aligned} \quad (23)$$

In terms of the dimensionless variables $\mathcal{J} = 4\beta J$ and $h = \beta H$, the free energy is

$$f(\bar{s}_A, \bar{s}_B) = \frac{-k_B T \log Z}{2NJ} = -\bar{s}_A\bar{s}_B - \mathcal{J}^{-1} \log [\cosh(\mathcal{J}\bar{s}_A - h)] - \mathcal{J}^{-1} \log [\cosh(\mathcal{J}\bar{s}_B - h)]. \quad (24)$$

The mean field equations are given by $\frac{\partial f}{\partial \bar{s}_A} = 0$ and $\frac{\partial f}{\partial \bar{s}_B} = 0$:

$$\begin{aligned} \bar{s}_A &= \tanh(h - \mathcal{J}\bar{s}_B), \\ \bar{s}_B &= \tanh(h - \mathcal{J}\bar{s}_A). \end{aligned} \quad (25)$$

(c) For $H = 0$, the solution to Eq. (25) has $\bar{s}_A = -\bar{s}_B = \bar{s}$ with

$$\bar{s} = \tanh(\mathcal{J}\bar{s}). \quad (26)$$

This is the same as the ferromagnetic case $J > 0$. The critical temperature is $T_c = 4J$ and below T_c we have $m \sim (T_c - T)^{1/2}$. Indeed, for $H = 0$ we can transform to the new variables

$$s'_i = \begin{cases} +s_i & i \in A \\ -s_i & i \in B \end{cases}, \quad (27)$$

so $s'_i s'_j = -s_i s_j$ and the sign of the exchange constant J in the Hamiltonian is reversed. Therefore, in this case the Ising antiferromagnet and ferromagnetic are equivalent. Note that this is not true for $H \neq 0$ because in terms of the s'_i the field would be reversed on the B sublattice. Also, the equivalence does not hold if we have next nearest neighbor interactions or are on a lattice that is not bipartite, e.g., the triangular lattice. In this case, the system is frustrated, meaning there is no configuration of spins that can simultaneously minimize the energy of all the interactions. Such systems will have finite entropy even at $T = 0$.

(d) For $H \neq 0$, it is helpful to introduce the new variables $\bar{s} = \frac{1}{2}(s_A + s_B)$ and $m = s_A - s_B$, so $s_{A,B} = \bar{s} \pm \frac{1}{2}m$. We expand Eq. (25) to third order in \bar{s} , m , and h :

$$\bar{s} \pm \frac{1}{2}m = h - \mathcal{J}(\bar{s} \mp \frac{1}{2}m) - \frac{1}{3} [h - \mathcal{J}(\bar{s} \mp \frac{1}{2}m)]^3 + \dots \quad (28)$$

Adding and subtracting the two equations, we obtain

$$\bar{s} = (h - \mathcal{J}\bar{s}) \left[1 - \frac{1}{3}(h - \mathcal{J}\bar{s})^2 - m^2 \right], \quad (29)$$

$$m = \left[\mathcal{J} - (h - \mathcal{J}\bar{s})^2 \right] m - \frac{1}{12}m^3. \quad (30)$$

From Eq. (29), we find to leading order

$$\bar{s} = \frac{h}{1 + \mathcal{J}}. \quad (31)$$

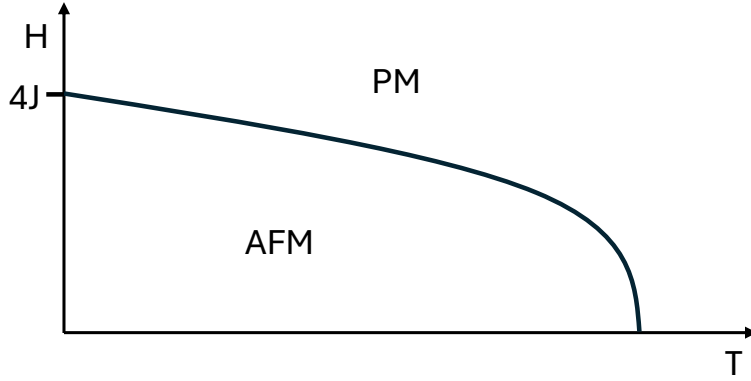


Figure 2: Phase diagram for the Ising antiferromagnet on a square lattice.

(e) Substituting Eq. (31) into Eq. (30), we have

$$[\mathcal{J} - 1 - h^2/(1 + \mathcal{J})^2] m - \frac{1}{12}m^3 = 0. \quad (32)$$

There is a phase transition when the coefficient of the term linear in m changes sign. For $h = 0$, this occurs at $\mathcal{J}_c = 1$. To leading order in h , $\mathcal{J}_c = 1 + \frac{1}{4}h^2$, so the critical temperature is

$$T_c(H) = T_c(H = 0)(1 - \frac{1}{4}h^2). \quad (33)$$

Near T_c , we have $\mathcal{J} = \mathcal{J}_c(1 - t)$ where $t = (T - T_c)/T$. For $t < 0$,

$$m = \pm \sqrt{-12J_c t}. \quad (34)$$

(f) A sketch of the phase diagram is shown in Fig. 2. For sufficiently low temperature and magnetic field, the system is in the antiferromagnetic phase (AFM), with $m > 0$. Upon increasing the temperature or field there is a phase transition to the paramagnetic phase (PM), with $m = 0$. Since m increases continuously from zero across the phase boundary, the transition is continuous everywhere except at $T = 0$, where it is first order. This is quite different from the ferromagnetic case, where for finite field there is no phase transition but rather a crossover with decreasing temperature. The origin of the difference is that for the ferromagnet the interaction and the field both try to align the spins, whereas for the antiferromagnet they are competing effects: the interaction tries to have neighboring spins anti-aligned while the field tries to align the spins.

3. Isotropic ferromagnet

(a) Analogously to the previous problem, we write $\vec{\sigma}_i = \vec{m} + \delta\vec{\sigma}_i$ and expand the Hamiltonian to first order in $\delta\vec{\sigma}_i$. We find the mean-field Hamiltonian

$$\beta\mathcal{H}_{\text{MF}} = \frac{1}{2}\beta N q J m^2 - \beta \sum_i (q J \vec{m} + \vec{H}) \cdot \vec{\sigma}_i = \frac{1}{2}N \mathcal{J} m^2 - \sum_i \vec{x} \cdot \vec{\sigma}_i. \quad (35)$$

The partition function for a single spin is

$$Z_1 = \int d\vec{\sigma} \exp \left[-\frac{1}{2}\mathcal{J}m^2 + \vec{x} \cdot \vec{\sigma} \right] = 2\pi e^{-\frac{1}{2}\mathcal{J}m^2} \int_{-1}^1 d \cos \theta e^{x \cos \theta} = \frac{4\pi e^{-\frac{1}{2}\mathcal{J}m^2} \sinh x}{x}, \quad (36)$$

and the free energy is

$$f(\vec{m}) = \frac{-k_B T \log Z_1}{qJ} = \frac{1}{2}m^2 + \mathcal{J}^{-1} \log(\sinh x/x) = \frac{1}{2}m^2 + \mathcal{J}^{-1} \log(\sinh x/x). \quad (37)$$

The average magnetization is

$$\vec{m} = \frac{\int d\vec{\sigma} \vec{\sigma} \exp[\vec{x} \cdot \vec{\sigma}]}{\int d\vec{\sigma} \exp[\vec{x} \cdot \vec{\sigma}]} = \frac{\partial f}{\partial \vec{x}} = \left(\coth x - \frac{1}{x} \right) \frac{\vec{x}}{x}. \quad (38)$$

Equivalently, \vec{m} is determined by minimizing the free energy, so $\frac{\partial f}{\partial \vec{m}} = 0$.

(b) We expand the free energy as

$$f(\vec{m}) = \frac{1}{2}m^2 + \frac{1}{3}\vec{h} \cdot \vec{m} + \frac{1}{6}\mathcal{J}m^2 - \frac{1}{180}\mathcal{J}^3m^4 + \dots \quad (39)$$

The corresponding mean field equation is

$$\vec{m} = \frac{1}{3}\vec{h} + \frac{1}{3}\mathcal{J}\vec{m} - \frac{1}{45}\mathcal{J}^3m^2\vec{m} + \dots \quad (40)$$

(c) For $\vec{h} = 0$, we have

$$\left(\frac{1}{3}\mathcal{J} - 1 - \frac{1}{45}\mathcal{J}^3m^2 \right) \vec{m} = 0 \quad (41)$$

The critical temperature is determined by when the coefficient of the linear term in this equation vanishes. This happens at $\mathcal{J} = \mathcal{J}_c = 3$, so $T_c = \frac{1}{3}qJ$. This model has a smaller T_c compared to the Ising model because it has more degrees of freedom and hence higher entropy.

(d) Near T_c , we expand $\mathcal{J} = \mathcal{J}_c(1 - t)$ to obtain

$$\left(-t - \frac{3}{5}m^2 \right) \vec{m} = 0. \quad (42)$$

For $t > 0$, we have $m = 0$, while for $t < 0$, we have

$$m = \pm \sqrt{-\frac{5}{3}t}. \quad (43)$$

Along T_c , the linear term in Eq. (40) vanishes, so

$$m \sim h^{1/3}. \quad (44)$$

(e) Since \vec{m}_0 and $\delta\vec{m}$ are along the same direction, $m^3 = (m_0 + \delta m_z)^3 = 3m_0^2\delta m_z$ near T_c .

With $\mathcal{J} = \mathcal{J}_c(1 - t)$, Eq. (40) becomes

$$\frac{1}{3}\delta h_z - \frac{1}{3}t\delta m_z - \frac{9}{5}m_0^2\delta m_z = 0, \quad (45)$$

hence,

$$\chi^L = \frac{\delta m_z}{\delta h_z} = \frac{1}{3(t - \frac{9}{5}m_0^2)} = \begin{cases} \frac{1}{3t} & t > 0 \\ \frac{1}{6|t|} & t < 0 \end{cases}. \quad (46)$$

- (f) In this case \vec{m}_0 and $\delta\vec{m}$ are in orthogonal directions, so $m^2\vec{m} = m_0^2\delta m_x\hat{x}$ near T_c . Eq. (40) becomes

$$\frac{1}{3}\delta h_x - \frac{1}{3}t\delta m_x - \frac{3}{5}m_0^2\delta m_x = 0, \quad (47)$$

and,

$$\chi_T = \frac{\delta m_x}{\delta h_x} = \frac{1}{3(t - \frac{3}{5}m_0^2)} = \begin{cases} \frac{1}{3t} & t > 0 \\ \infty & t < 0 \end{cases}. \quad (48)$$

Below T_c , the applied field h_0 breaks the $O(n)$ symmetry of the model and makes \vec{m} point along the z direction. The longitudinal susceptibility corresponds to a change in the magnitude of \vec{m} , whereas the transverse susceptibility corresponds to a change in the direction of \vec{m} . Changing the direction of \vec{m} is a zero energy mode of the system known as a Goldstone mode, hence the infinite χ_T .

4. Potts model

- (a) The number of configurations is

$$\Omega = \frac{N!}{N_1!N_2!N_3!}. \quad (49)$$

The entropy is

$$S(x_1, x_2, x_3) = \ln \Omega = N \ln N - N \sum_{s=1}^3 (N_s \ln N_s - N_s) = -N \sum_{s=1}^3 x_s \ln x_s. \quad (50)$$

- (b) The average energy is

$$E(x_1, x_2, x_3) = -J \sum_{\langle ij \rangle} \sum_{s_i=1}^3 [\hat{\rho}_i(s_i)\hat{\rho}_j(s_j) - 1] = -\frac{1}{2}NcJ \sum_{s=1}^3 (3x_s^2 - 1), \quad (51)$$

with c the coordination number of the lattice ($c = 4$ for the square lattice).

- (c) We have

$$x_1 = \frac{1}{3}(1 + 2m), \quad x_{2,3} = \frac{1}{3}(1 - m). \quad (52)$$

- (d) The mean field free energy is

$$\begin{aligned} \frac{\psi}{Nk_B T} &= -\frac{cJ}{2k_B T} \sum_{s=1}^3 (3x_s^2 - 1) + \sum_{s=1}^3 x_s \ln x_s \\ &= -\frac{cJ}{k_B T} m^2 + \frac{1}{3} [(1 + 2m) \ln(1 + 2m) + 2(1 - m) \ln(1 - m)] \\ &= -\frac{T - T_0}{T} m^2 - \frac{1}{3} m^3 + \frac{1}{2} m^4 + \dots, \end{aligned} \quad (53)$$

with $T_0 = cJ/k_B$. We made the mean-field approximation by taking the distribution $\hat{\rho}$ to have the factorized form $\hat{\rho}(s_1, \dots, s_N) = \hat{\rho}_1(s_1) \dots \hat{\rho}_N(s_N)$. This treats each spin independently and ignores correlations between them.

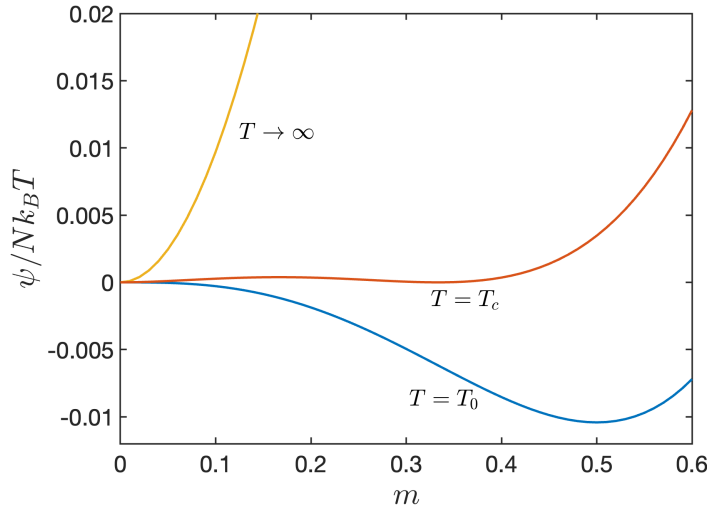


Figure 3: Free energy ψ versus m at different temperatures.

(e) Let $\alpha = (T - T_0)/T$, so

$$\frac{\psi(m)}{Nk_B T} = \alpha m^2 - \frac{1}{3}m^3 + \frac{1}{2}m^4, \quad (54)$$

To find the extrema of $\psi(m)$, we solve

$$0 = \frac{\psi'(m)}{Nk_B T} = 2\alpha m - m^2 + 2m^3, \quad (55)$$

which has the three roots $m = 0$ and

$$m_{\pm} = \frac{1}{4} (1 \pm \sqrt{1 - 16\alpha}). \quad (56)$$

Thus, for $\alpha > \frac{1}{16}$ there is only one real solution, $m = 0$, while for $\alpha < \frac{1}{16}$ there are three real solutions. To find which solution has the lowest free energy, we set $\psi(m) = \psi(0) = 0$, which means m must satisfy the equations

$$\begin{aligned} 2\alpha_c - m + 2m^4 &= 0, \\ \alpha_c - \frac{1}{3}m + \frac{1}{2}m^2 &= 0. \end{aligned} \quad (57)$$

Eliminating the quadratic term gives $m = 6\alpha_c$, and substituting $m = m_+$ we find $\alpha_c = \frac{1}{18}$. Therefore, there is a phase transition at the critical temperature

$$T_c = \frac{18}{17}T_0 \simeq 1.059T_0. \quad (58)$$

For $T < T_c$, $m = m_+$ (and in particular $m = \frac{1}{2}$ for $T = T_0$), whereas for $T > T_c$, $m = 0$. Since m changes discontinuously across T_c , this is a first order phase transition, which is generic for mean field free energies that contain a cubic term (and true for the Potts model with $q > 2$). In Fig. 3, we plot $\psi(m)$ for the three cases $T = T_0$, $T = T_c$, and $T \gg T_0$.