

PHYSICS 210A – Spring 2024

Problem Set #1 Thermodynamics & Microcanonical Ensemble

Due date: Monday, April 15

- 1. Review of the heat engine.** Consider the Carnot cycle applied to an ideal gas of N particles. Starting at temperature T_2 and volume V_a , heat is injected into the system to drive isothermal expansion to a volume $V_b > V_a$. This is followed by adiabatic expansion until the temperature drops to T_1 at which point the volume reaches $V_c > V_b$. Then with isothermal compression at temperature T_1 , the volume is reduced to $V_d < V_c$. Finally the volume is compressed adiabatically back to the starting volume $V_a < V_d$, while temperature is increased back to T_2 .
 - a)** From the equation of state for an ideal gas, $PV = Nk_B T$, and the additional constraint that $PV^{5/3} = \text{const}$ for an adiabatic ideal gas, compute the work done by the system for each of the four legs of the cycle, $W_{ab}, W_{bc}, W_{cd}, W_{da}$. By convention, work done by a system is positive.
 - b)** Next, we compute the heat absorbed by the system for each leg of the cycle, $Q_{ab}, Q_{bc}, Q_{cd}, Q_{da}$. By convention, heat absorbed by a system is positive. [Hint: Use the first law of thermodynamics, and exploit the fact that the energy of an ideal gas, $E = \frac{3}{2}Nk_B T$, is independent of the volume.]
 - c)** Show that the volumes V_c, V_d are slaved to the choices of V_a, V_b , and hence express your answers to parts a) and b) in term of the ratio $V_b : V_a$.
 - d)** Show that the efficiency of the Carnot engine, η , defined as the ratio of the total work done by the system to the heat absorbed by the system over a cycle, is independent of the expansion ratio $V_b : V_a$ and only on the temperature ratio $T_1 : T_2$.

Next calculate the efficiency of the Stirling cycle, which is similar to the Carnot cycle, except that the two adiabatic processes of the Carnot cycle are replaced by two isochoric processes, i.e., at the end of isothermal expansion at temperature T_2 , the system is cooled to temperature T_1 at fixed volume $V_c = V_b$, and at the end of isothermal compression at temperature T_1 , the system is heated back to temperature T_2 at fixed volume $V_d = V_a$.

[Hint: Recall that the specific heat of an ideal gas at constant volume is $C_V = \frac{3}{2}Nk_B$.]

- 2. Maxwell-Boltzmann Distribution.** In class, we wrote down the following expression for the density of state $\Omega(E, V, N)$ which defined the microcanonical ensemble for an ideal gas,

$$\Omega(E, V, N) = \frac{1}{h^{3N}} \int_V d^3\vec{r}_1 \dots d^3\vec{r}_N \int d^3\vec{p}_1 \dots d^3\vec{p}_N \delta\left(E - \frac{p_1^2 + \dots + p_N^2}{2m}\right).$$

- a)** Write down a similar expression defining the probability density $\mathcal{P}(\vec{p})$ of finding a particle with momentum \vec{p} within a “volume” $d^3\vec{p}$ in this ensemble.
- b)** Express $\mathcal{P}(\vec{p})$ in term of $\Omega(E)$, and work out its dependence on E, V, N using the expression for $\Omega(E)$ worked out in class.
- c)** In the limit of large N , use Stirling’s formula $\ln N! \approx N \ln N - N$ to simplify your result for $\mathcal{P}(\vec{p})$. Show your expression is properly normalized.
- d)** Write down the probability density $\mathcal{P}(p)$ of finding the absolute momentum $p \equiv |\vec{p}|$ within a small momentum bin dp . Use the relation between E and T to express $\mathcal{P}(p)$ in term of T to display its familiar form, the Maxwell-Boltzmann momentum distribution.

- 3. Gibbs factor.** Suppose the Gibbs factor is not included in the expression for entropy, i.e., suppose we take $S(E) = k_B \ln \Omega(E)$. Work out the entropy $S(E, V, N)$ for the ideal gas in the microcanonical ensemble, and from it, find the temperature T , pressure P , and chemical potential μ as functions of the state variables E, V, N . Explain the problem(s) when Gibbs factor is not included.
- 4. Specific heat of a solid.** Consider a solid having N atoms. Each atom i is modeled as a harmonic oscillator, with momentum \vec{p}_i and coordinate \vec{q}_i , with the latter defined with respect to the equilibrium position of the oscillator, arranged in a 3D lattice. This system is thus described by the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^N \left[\frac{p_i^2}{2m} + \frac{m\omega^2 q_i^2}{2} \right]$$

where each oscillator is distinguished by its distinct equilibrium position.

a) Calculate the entropy S as a function of the total energy E directly from the microcanonical ensemble.

[Hint: the integral you encounter can be converted to a spherical integral.]

b) Calculate the temperature T in terms of E and N . Calculate the heat capacity $C = \left(\frac{\partial E}{\partial T} \right)_N$.

This result is the law of Dulong-Petit.