Physics 210A — Spring 2024
Practice problem solutions

1. **Hard-core particles on a line**

   (a) The Hamiltonian is

$$\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2m} + f x_N = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + f a_i \right), \tag{1}$$

   where $a_i = x_i - x_{i-1}$ is the separation distance of adjacent beads. The partition function may be determined by integrating over the a_i's. Since the beads are impenetrable with diameter a, we must have $a_i > a$. Therefore,

$$Z(\beta, f, N) = \frac{1}{h^N} \prod_{i=1}^{N} \int dp_i \int_a^\infty da_i \, e^{-\beta\mathcal{H}}, \tag{2}$$

   and $Z = Z_1^N$ where

$$Z_1(\beta, f) = \frac{1}{h} \int dp \int_a^\infty dx \, \exp\!\left[-\beta\left(\frac{p^2}{2m} + fx\right)\right] = \frac{1}{h}\sqrt{\frac{2\pi m}{\beta}} \frac{e^{-\beta f a}}{\beta f}. \tag{3}$$

   (b) The energy is

$$E = -\frac{\partial \ln Z}{\partial \beta} = N(\tfrac{3}{2} k_B T + fa), \tag{4}$$

   and the specific heat is

$$C_f = \frac{\partial E}{\partial T} = \tfrac{3}{2} N k_B. \tag{5}$$

   Furthermore,

$$\langle x_N \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial f} = N(k_B T / f + a). \tag{6}$$

   Note the factor $\frac{3}{2}$ in the heat capacity although the system is one-dimensional. This is because we are using a constant force ensemble, analogous to constant pressure. If we Legendre transform $\ln Z$ to find the free energy at constant length, we may obtain the heat capacity at constant length $\frac{1}{2} N k_B$, in accordance with the equipartition theorem (show this!).

   (c) From Eq. (6), we find the equation of state

$$f = \frac{n k_B T}{1 - na}. \tag{7}$$

   The term na in the denominator account for the excluded volume due to the finite diameter of the beads, as in the van der Waals equation of state. The equation of state reduces to an ideal gas in the dilute limit $na \ll 1$.

2. **Array of magnetic moments**

   The partition function for a single atom is

$$Z_1(\beta) = \sum_{\sigma=\pm 1} e^{-\beta \mu H \sigma} = 2\cosh(\beta \mu H), \tag{8}$$

and the total partition function is $Z = Z_1^N$. The mean energy is

$$E = -\frac{\partial \ln Z}{\partial \beta} = -N\mu H \tanh(\beta\mu H), \tag{9}$$

and the entropy is

$$S = \ln Z + \beta E = N \ln[2\cosh(\beta\mu H)] - N\beta\mu H \tanh(\beta\mu H). \tag{10}$$

In the limit $H \to 0$, the entropy is $N \ln 2$, meaning that the system is equally likely to be in any of the 2^N microstates. In the limit $H \to \infty$, the entropy vanishes since all atoms are spin-polarized along the field direction. The energy variance is

$$\sigma_E^2 = \frac{\partial^2 \ln Z}{\partial \beta^2} = -\frac{\partial E}{\partial \beta} = N\mu^2 H^2 \operatorname{sech}^2(\beta\mu H). \tag{11}$$

3. **Surface fluctuation**

   (a) The partition function for a single step is

   $$Z_1(\beta) = \sum_{\sigma=0,\pm 1} e^{-\beta\varepsilon(1+\sigma^2)} = e^{-\beta\varepsilon}(1 + 2e^{-\beta\varepsilon}). \tag{12}$$

   Since each step is independent of the others, the total partition function is $Z = Z_1^N$.

   (b) We have

   $$F = -k_B T \ln Z = N[\varepsilon - k_B T \ln(1 + 2e^{-\beta\varepsilon})]. \tag{13}$$

   At low temperature, $\beta\varepsilon \ll 1$, the free energy reduces to $N\varepsilon$, meaning the surface becomes flat to minimize its energy. At high temperature, $\beta\varepsilon \gg 1$, the free energy is $-Nk_B T \ln 3$. It is dominated by the entropy, which attains the maximum value $Nk_B \ln 3$, since the system is equally likely to be in any of the 3^N microstates [see Fig. 1(a)].

   (c) The total length of the surface is

   $$L = \left\langle \sum_{i=1}^{N} (1 + \sigma_i^2) \right\rangle = -\frac{\partial \ln Z}{\partial \beta\varepsilon} = N\left[1 + \frac{2}{1 + 2e^{\beta\varepsilon}}\right]. \tag{14}$$

   We have $L = N$ at low temperature and $L = \frac{5}{3}N$ at high temperature [see Fig. 1(b)].

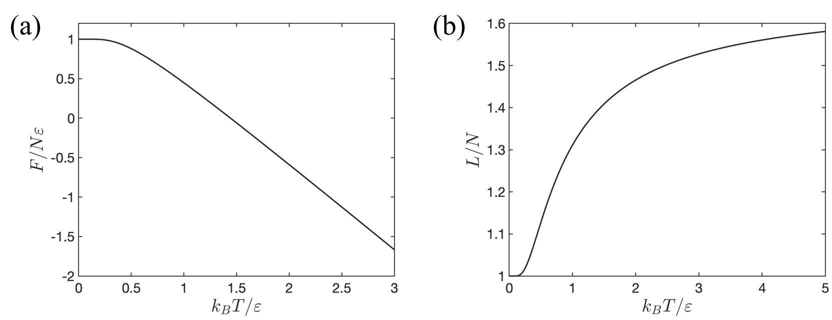

Figure 1: (a) Free energy of surface versus temperature. (b) Length of surface versus temperature.