Phys 210A — Spring 2024

Practice Problems

1. **Hard-core Particles on a Line** — A system of $N + 1$ impenetrable beads of diameter a are placed on a frictionless wire. The left most bead is fixed at the origin, i.e., $x_0 = 0$, and the right most bead is subject to a constant force f toward the origin.

   (a) Calculate the partition function of the system as a function of N, β, and f, to leading order in a assuming that $N \gg 1$.

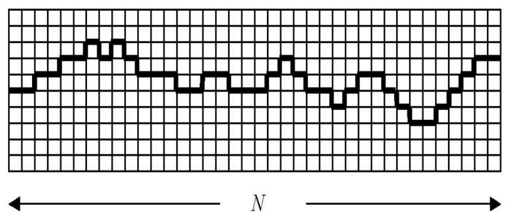
   (b) Calculate the specific heat C_f of the system and the average position $\langle x_N \rangle$ of the rightmost bead as functions of T.

   (c) For this one-dimensional toy system, the "pressure" P is simply the force being applied to the last particle. Find the equation of state $P(n, T)$, where $n \equiv N/\langle x_N \rangle$ is the one dimensional density. In which limit does the equation of state reduce to that of an ideal gas ?

2. **Array of magnetic moments** — The magnetic moment of an atom may take two orientations with respect to an external magnetic field H: aligned with the field, with energy $-\mu H$, or against the field with energy $+\mu H$, where μ is the Bohr magneton. Find the mean energy of N such atoms, and evaluate the entropy in the limits $H \to 0$ and $H \to \infty$. Show that the standard deviation of the energy of the array is

$$\sigma_E = \mu H N^{1/2} \mathrm{sech}(\beta \mu H).$$

3. **Surface fluctuation** — The figure below illustrates a simple model of the surface of a two-dimensional 'solid' confined to a square lattice. The two ends of the surface are N lattice sites apart, with $N \gg 1$. The surface energy is proportional to the surface length, with energy $\varepsilon > 0$ per lattice length. The surface height can change by at most one lattice length at a time. (Overhangs are forbidden, so that outward-pointing surface normals never point downward.)



Thus the surface can be modeled by a Hamiltonian

$$\mathcal{H} = \varepsilon \sum_{i=1}^{N} (1 + \sigma_i^2),$$

where $\sigma_i = +1$, 0, or -1 depending on whether the i^{th} 'column' contains a step up, no step, or step down, respectively.

(a) Find the partition function of the surface at temperature T.

(b) Sketch the temperature dependence of the free energy $F(T)$ of the surface. Interpret your result physically in the limit $k_B T \ll \varepsilon$ and $k_B T \gg \varepsilon$.

(c) Find the total length of the surface (including all the up and down steps) as a function of the temperature and sketch its temperature dependence.