Phys 210A — Spring 2024
Mock Final Exam

`You are allowed to bring up to 5 handwritten sheets of notes. Copies of handwritten sheets are okay, but not copies of anything not written by you, including lecture notes. Turn in the notes you used along with your exam.`

`The point assignment is the suggested time to spend on each problem.`

1. **Short Questions** `[30/180 min]`

   (a) Explain the two key ingredients that underly van der Waals' model of a "real" gas.

   (b) Explain why $dP/dV > 0$ implies an instability.

   (c) Explain the difference in the physical origin of the thermodynamic singularity associated with first-order and second-order phase transitions.

   (d) Explain why the mean-field theory describes the Ising model exactly as the spatial dimension d approaches infinity.

   (e) Explain why there cannot be long-range order for the one-dimensional Ising model at any finite temperature.

   (f) What is the temperature dependence of the specific heat of an ideal fermi gas at very low temperatures (i.e., where $k_B T$ is much smaller than the Fermi energy ε_F). Explain qualitatively how this temperature dependence arises.

2. **Confinement cost of a polymer** `[20/180 min]` — In class, we showed that a Gaussian chain of L links of microscopic size a_0 has an expected end-to-end distance $R_0 = a_0 \cdot L^{1/2}$, and the free energy cost of confining this system to a box of size W with $a_0 \ll W \ll R_0$ is

$$\Delta F_{\text{confine}} \propto \frac{k_B T L}{(W/a_0)^2}.$$

   (a) Explain the physical origin of this confinement cost.

   (b) A self-avoiding polymer is larger than the non-interacting Gaussian chain in the long length ($L \gg 1$) limit, with the expected end-to-end distance being $R = a \cdot L^{\nu}$, where a is a microscopic size and the exponent ν is between 1/2 and 1. Find how the confinement cost scales with W in term of the exponent ν for the self-avoiding polymer.
   `[Note: no formal calculation is needed here. You are asked to provide an estimate and describe how you think through the problem.]`

3. **Quantum ideal gas** `[20/180 min]` – Show that in d spatial dimensions, the average energy $E \equiv \langle \mathcal{H} \rangle$ of a quantum ideal gas with the Hamiltonian $\mathcal{H} = p^2/(2m)$ is given by $E = c \cdot PV$. Find how the constant c depends on d for bosons and fermions. Based on your solution, what can you say about classical ideal gas obeying Boltzmann statistics?
`[Hint: An integral that arises can be simplified by partial integration.]`

4. **Vibrational Specific Heat of Solids** `[50/180 min]` — In this problem, we will analyze the contribution to the specific heat of a crystalline solid due to its quantized lattice vibrations, the *phonons*.

   In the Einstein model of lattice vibration, one approximates a lattice of N atoms by noninteracting (quantum) harmonic oscillators, each with frequency ω. Three oscillators per atom are used to describe motion in the x, y, z directions.

   (a) Compute the partition function of this system, and hence obtain the mean energy $\bar{\varepsilon}(T;\omega)$ per atom.

   (b) Find the specific heat $C_V(T)$. Show that in the limit $T \to \infty$, the *Dulong-Petit law* of classical physics is recovered. Show also that $C_V(T) \to 0$ as $T \to 0$.

   The explanation of vanishing specific heat at low T, as a result of quantum mechanics, was a great triumph of the Einstein theory. A more quantitatively accurate theory was due to Debye, who assumed that each oscillator can take on a range of frequencies ω_i, corresponding to the different normal mode $\mathbf{k}$ of lattice vibration. Alternatively, one can view the quantized lattice vibrations (phonons) as an ideal bose gas, with energies $\varepsilon(\mathbf{k}) = \hbar\omega(\mathbf{k})$. It is useful to introduce the frequency distribution function $\mathcal{D}(\omega)$, where $\mathcal{D}(\omega)d\omega$ describes the number of modes having frequency between ω and $\omega + d\omega$.

   (c) What is the mean energy per atom $\bar{\varepsilon}(T)$ in term of $\mathcal{D}(\omega)$? Find $\bar{\varepsilon}(T)$ in the limit of high temperature, thereby showing that the high temperature behavior is indepedent of the choice $\mathcal{D}(\omega)$. `[Note the normalization condition on` $\mathcal{D}(\omega)$ `is` $\int_0^\infty d\omega \mathcal{D}(\omega) = 3N$`.]`

   (d) Low temperature behavior does depend on the form of $\mathcal{D}(\omega)$, which can be obtained from the knowledge of the phonon *dispersion relation* $\omega(\mathbf{k})$. Assuming that $\omega(\mathbf{k}) = v \cdot |\mathbf{k}|$, where v is the sound speed, show that $\mathcal{D}(\omega) = \alpha V \omega^2$, with V being the volume of the solid and α being a proportionality constant.

   To satisfy the normalization condition on $\mathcal{D}(\omega)$, Debye introduced an approximation in which he took the result of (d) up to some cutoff frequency ω_D, chosen to give a total of $3N$ modes. For $\omega > \omega_D$, Debye assumed that $\mathcal{D}(\omega) = 0$.

   (e) Using Debye's approximation, write down an expression for the mean energy per atom $\bar{\varepsilon}$ in terms of $\hbar\omega_D$ and a dimensionless parameter T_D/T, where the *Debye temperature* T_D is defined by $k_B T_D = \hbar\omega_D$.

   (f) Find the vibrational specific heat $C_V(T)$ for $T \ll T_D$. Show that the T-dependence follows generally from the $k \to 0$ limit of the dispersion relation and is insensitive to Debye's approximation.

5. **Isotropic ferromagnet** `[60/180 pt]` — An isotropic ferromagnet may be described by the Heisenberg model,

$$\beta\mathcal{H} = -\beta J \sum_{\langle i,j \rangle} \vec{\sigma}(i) \cdot \vec{\sigma}(j) - \beta \vec{H} \cdot \sum_i \vec{\sigma}(i),$$

   where $\vec{\sigma}(i)$ is a 3d unit vector giving the direction of a classical spin vector located on lattice site i, $J > 0$ is the strength of the ferromagnetic interaction energy, $\vec{H} \cdot \vec{\sigma}(i)$ gives the magnetic energy of the spin at site i in an uniform external magnetic field $\vec{H}$, and $\langle i, j \rangle$ denotes nearest neighbor sites.

(a) Apply the mean-field approximation and write down the mean-field Hamiltonian $\mathcal{H}_{\mathrm{MF}}$ for the system. From the mean-field Hamiltonian, derive a self-consistent equation of state describing how the average magnetization per site $\vec{m} \equiv \langle \vec{\sigma} \rangle$ depends on $\vec{h} \equiv \beta \vec{H}$ and on T. You may take this system to have N sites with each site having q nearest neighbors.]
`[Hint: You may also assume the average magnetization` $\vec{m}$ `to point in the direction of the field` $\vec{H}$`, and express this problem in the form of an isolated spin in an effective magnetic field. The latter is a problem we worked out during the first half of the class.]`

(b) Find the critical temperature T_c in the absence of the external field (i.e., for $\vec{h} = 0$) by expanding the self-consistent equation of state in the vicinity of T_c. Explain why the numerical value is different from $T_c = qJ/k_B$ found for the Ising model. Obtain the dependence of the magnetization on $t = (T - T_c)/T_c$ in the vicinity of the critical temperature.
`[Taylor expansion:` $\cosh(x) \approx 1 + x^2/2$ `and` $\sinh(x) \approx x + x^3/6$`.]`

(c) For small $\vec{h}$ and small $|t|$, obtain the approximate form of the equation of state to important leading order in $\langle \vec{\sigma} \rangle$. How does $\langle \vec{\sigma} \rangle$ depend on $\vec{h}$ at $t = 0$?

(d) Suppose the applied field is pointed in the z-direction. Find $\frac{\partial \langle \sigma_z \rangle}{\partial h_z}$ for small $|t|$ and h_z, and from it, the "longitudinal susceptibility",

$$\chi_L(t) = \lim_{h_z \to 0} \frac{\partial \langle \sigma_z \rangle}{\partial h_z}$$

for $t > 0$ and $t < 0$. `[Here,` σ_z `and` h_z `refer to the` z`-component of` $\vec{\sigma}$ `and` $\vec{h}$ `respectively.]`

(e) Suppose there is additionally an infinitesimal field $h_x \ll h_z$ in the x-direction, write down the relation between $\langle \sigma_x \rangle$ and h_x. Find $\left. \frac{\partial \langle \sigma_x \rangle}{\partial h_x} \right|_{h_x = 0}$ for small $|t|$ and h_z, and from it the "transverse susceptibility"

$$\chi_T(t) = \lim_{h_z \to 0} \lim_{h_x \to 0} \frac{\partial \langle \sigma_x \rangle}{\partial h_x}$$

for $t > 0$ and $t < 0$. Can you explain qualitatively why χ_T behaves so differently from χ_L below T_c?