Physics 210A — Spring 2024
Midterm solutions

1. **Random walk**

We are given $\langle a_{n,y}^2 \rangle = \frac{2}{3}a^2$. Since the steps have fixed length a,

$$a^2 = \langle |\vec{a}_n|^2 \rangle = \langle a_{n,x}^2 \rangle + \langle a_{n,y}^2 \rangle, \tag{1}$$

which implies $\langle a_{n,x}^2 \rangle = \frac{1}{3}a^2$. The steps are uncorrelated with each other, so $\langle a_{n,x} a_{m,x} \rangle = \frac{1}{3}a^2\delta_{nm}$ and $\langle a_{n,y} a_{m,y} \rangle = \frac{2}{3}a^2\delta_{nm}$. Let $\vec{R} = \sum_{n=1}^N \vec{a}_n$ be the position vector after N steps. We have

$$\langle R_x^2 \rangle = \left\langle \left(\sum_{n=1}^N a_{n,x} \right)^2 \right\rangle = \sum_{n,m=1}^N \langle a_{n,x} a_{m,x} \rangle = \sum_{n,m=1}^N \tfrac{1}{3} a^2 \delta_{nm} = \tfrac{1}{3} N a^2, \tag{2}$$

and similarly $\langle R_y^2 \rangle = \frac{2}{3}Na^2$. Therefore, after N steps the RMS distances in the x and y directions are

$$\Delta x = \sqrt{\langle R_x^2 \rangle} = \sqrt{N/3}\, a, \ \Delta y = \sqrt{\langle R_y^2 \rangle} = \sqrt{2N/3}\, a. \tag{3}$$

2. **Integer spin**

The partition function for a single spin is

$$Z_1 = \sum_{s=0,\pm 1} e^{\beta H s} = 1 + 2\cosh(\beta H), \tag{4}$$

and the total partition function is $Z = Z_1^N$. The magnetization is

$$M = \frac{\partial \ln Z}{\partial(\beta H)} = N \frac{2\sinh(\beta H)}{1 + 2\cosh(\beta H)}. \tag{5}$$

Thus, we find for the zero-field susceptibility

$$\chi = \lim_{H\to 0} \frac{\partial M}{\partial H} = \lim_{H \to 0} N\beta \left[\frac{2\cosh(\beta H)}{1 + 2\cosh(\beta H)} - \frac{4\sinh^2(\beta H)}{[1 + 2\cosh(\beta H)]^2} \right] = \frac{2N}{3k_B T}. \tag{6}$$

3. **Connecting the micro-canonical and canonical ensembles**

We draw a tangent line to the plot with slope

$$\frac{\partial S}{\partial E} = \frac{1}{T} = \frac{1}{300\,\mathrm{K}} \approx 0.4 \frac{8.3\,\mathrm{J/K}}{1\,\mathrm{kJ}}, \tag{7}$$

as shown in Fig. 1. The energy at the tangent point is the average energy $\bar{E} = 4.5$ kJ, and $S(\bar{E}) = 8.3$ J/K$=N_A k_B$. The free energy is the x-intercept of the tangent line: $F = \bar{E} - TS(\bar{E}) = 2$ kJ. Since $S = k_B \ln \Omega$, where Ω is the number of microstates at the given energy, the number of states at energy $\bar{E}$ is

$$\Omega(\bar{E}) = e^{S(\bar{E})/k_B} = e^{N_A} = e^{6.02 \times 10^{23}}. \tag{8}$$

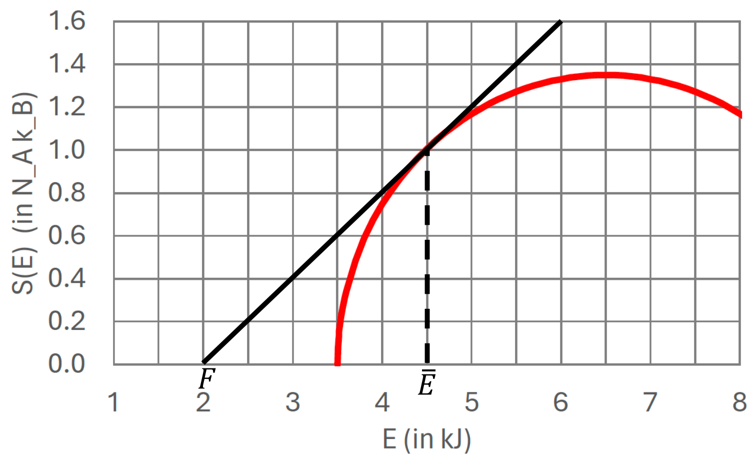

Figure 1: Finding the average energy $\bar{E}$ and free energy F from $S(E)$ at $T = 300$ K.