PHYSICS 210A – Spring 2022
Problem Set #1 Microcanonical Ensemble
Due date: Monday, April 11

1. In this problem, you will use the method of steepest descent to derive the Stirling formula, i.e., the large-$n$ limit of the gamma function

$$\Gamma(n + 1) = \int_0^{\infty} dx \ x^n e^{-x}$$

which is equal to $n!$ for integer $n$.

a) Write the integrand as $e^{f(x)}$ and show that it has a single maximum at some value $x^*$.

b) Justify that $f(x)$ can be approximated by a polynomial of $x - x^*$ to the 2nd order in the large-$n$ limit, and carry out that the integration in the large-$n$ limit.

c) What is the next order correction?

2. In class, we wrote down the following expression for the density of state $\Omega(E, V, N)$ which defined the microcanonical ensemble for an ideal gas,

$$\Omega(E) = \frac{1}{h^{3N}} \int_V d^3\vec{r}_1 ... d^3\vec{r}_N \int d^3\vec{p}_1 ... d^3\vec{p}_N \ \delta \left(E - \frac{p_1^2 + \cdots + p_N^2}{2m}\right).$$

a) Write down a similar expression defining the probability $P(\vec{p})$ of finding a particle with momentum $\vec{p}$ in this ensemble.

b) Express $P(\vec{p})$ in term of $\Omega(E)$, and work out its dependence on $E, V, N$ using the expression for $\Omega(E)$ worked out in class.

c) In the limit of large $N$, use Stirling’s formula $N! \approx N \ln N - N$ to simplify your result for $P(\vec{p})$. Show your expression is properly normalized.

d) Use the relation between $E$ and $T$ to express $P(\vec{p})$ in term of $T$ to display its familiar form, the Maxwell-Boltzmann distribution.

3. Suppose the Gibbs factor is not included in the expression for entropy, i.e., suppose we take $S(E) = k_B \ln \Omega(E)$. Work out the entropy $S(E, V, N)$ for the ideal gas in the microcanonical ensemble, and from it, find the temperature $T$, pressure $P$, and chemical potential $\mu$ as functions of the state variables $E, V, N$. Explain the problem(s) when Gibbs factor is not included.

4. Consider $N$ harmonic oscillators with coordinates and momentum $\{\vec{q}_i, \vec{p}_i\}$, and the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} \left[ \frac{p_i^2}{2m} + \frac{m \omega^2 q_i^2}{2} \right].$$

Take the coordinates to be given with respect to the equilibrium positions of each oscillator which is arranged in a 3D lattice, so that each oscillator is distinguished.

a) Calculate the entropy $S$ as a function of the total energy $E$.

[Hint: You will encounter a surface integral that is difficult to do exactly. But for $N \gg 1$, you may ignore the difference between surface and volume integral.]

b) Calculate the temperature $T$ in term of $E$ and $N$. Calculate the heat capacity $C = \left(\frac{\partial E}{\partial T}\right)_N$. 
