Problem 1

(a)
The probability distribution for a single throw is equal for each throw, giving us the expectation value:

\[ P(x_i) = \frac{1}{6}, \quad \langle X_N \rangle = N \sum_{i=1}^{6} x_i P(x_i) = \frac{N}{2}. \]  

(1)

This gives us the bias of the random walk, and is non-zero.

(b)
To find the variance after \( N \) steps, we write

\[ \langle X_N^2 \rangle = \sum_{j=1}^{N} \sum_{j'=1}^{N} \langle x_j x_{j'} \rangle, \]  

(2)

where the angled brackets denote an average over the \( N \)-step probability distribution \( P(x_1, ..., x_N) = \prod_{i=1}^{N} P(x_i) \). This average is

\[ \langle x_j^2 \rangle = \frac{91}{6}, \quad j = j', \]  

(3)

\[ \langle x_j x_{j'} \rangle = \langle x_j \rangle^2 = \frac{1}{4}, \quad j \neq j' \]  

(4)

using the expression for probability above. This can also be rewritten as

\[ \langle x_j x_{j'} \rangle = \frac{1}{4} + \delta_{jj'} \left( \frac{91}{6} - \frac{1}{4} \right), \]  

(5)

which gives us

\[ \langle X_N^2 \rangle = \langle X_N \rangle^2 + N \left( \frac{91}{6} - \frac{1}{4} \right), \quad \Delta_N^2 = \frac{179N}{12}. \]  

(6)
From this, the drift contribution is equal to the RMS fluctuation when
\[
\frac{N^*}{2} = \sqrt{\frac{179 N^*}{12}},
\]  
with the non-zero root at \( N^* = \frac{179}{3} \), or at step \( N^* = 60 \).

**Problem 2**

**(a)**

The energy required for the state with \( j \) links open is \( j \varepsilon \), since all of the links up to that point need to be open, each with a cost of \( \varepsilon \). The canonical partition function is obtained by summing over all states with the Boltzmann weight:
\[
Z = \sum_{n=0}^{N} e^{-j \beta \varepsilon} = \frac{1 - e^{-(N+1) \beta \varepsilon}}{1 - e^{-\beta \varepsilon}}.
\]  

**(b)**

The average length of the crack is proportional to the average energy \( \langle E \rangle = \varepsilon \langle L \rangle \), so we can find the average length of the crack the same way:
\[
\langle L \rangle = -\frac{1}{\varepsilon} \frac{\partial \ln Z}{\partial \beta} = \frac{1}{e^{\beta \varepsilon} - 1} - \frac{N + 1}{e^{\beta (N+1) \varepsilon} - 1}.
\]

**Problem 3**

**(a)**

We have three species of sites (\( N \) total sites) in this system: \( N_1, N_\downarrow, N_0 \), which correspond to sites with up, down and no spins adsorbed. The total number of ways to fill these sites is
\[
\Omega(N_1, N_\downarrow, N_0) = \binom{N}{N_1} \frac{N}{N_\downarrow} \frac{N}{N_0} = \frac{N!}{N_1! N_\downarrow! N_0!}.
\]  

We can write this expression in terms of \( Q, M = N_1 \pm N_\downarrow \):
\[
\Omega(Q, M, N) = \binom{N}{\frac{Q+M}{2}} \binom{N-M}{\frac{Q-M}{2}} \binom{N}{N-Q},
\]  

with the entropy given by \( S(Q, M) = k_B \ln(\Omega(Q, M, N)) \).

**(b)**

We can expand the entropy using Stirling’s approximation \( \ln N! \approx \ln N - N \). After some algebra, we obtain
\[
\frac{S}{Nk_B} \equiv s = -\frac{q + m}{2} \ln \frac{q + m}{2} - \frac{q - m}{2} \ln \frac{q - m}{2} - (1 - q) \ln(1 - q), q = \frac{Q}{N}, m = \frac{M}{N}.
\]  

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The temperature $T(q,m)$ can be found as the derivative of entropy with respect to energy. Recalling $E = -\alpha Q$, we find
$$\frac{1}{k_B T(q,m)} = -\frac{1}{\alpha} \frac{\partial s}{\partial q} = \frac{1}{\alpha} \left( 1 - \ln 4 + \ln \frac{q^2 - m^2}{1 - q} \right). \quad (13)$$

(c)

We can solve for the zero of inverse temperature $\beta$ to find $q^* = \frac{2}{e} \left( \sqrt{1 + e + \frac{m^2 e^2}{4}} - 1 \right)$. Below this value, the temperature becomes negative. Negative temperature means that a gain in energy gives a decrease in entropy. For this problem, if $q$ is small enough, then adsorbing more particles reduces the entropy of the system by reducing the number of vacancies, and, therefore, the entropy.

**Problem 4**

The occupancy of the $n$th level of a Bose gas is given by (see next problem for a detailed derivation)
$$n_B(\varepsilon_k) = \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} \quad (14)$$

with $\mu$ being the chemical potential of the gas and $\varepsilon_k$ being the energy of that state at wavevector $k$. We do not expect macroscopic occupancy of the excited states, since the occupation of the ground state is not limited, unlike the occupation of the excited states in dimensions $d > 2$.

**Problem 5**

The phonon dispersion $\varepsilon_k = \hbar ck$, with a chemical potential $\mu = 0$. The reason a photon gas has no chemical potential is because their number is fixed by the thermal equilibrium between the gas and the external heat reservoir. Recall that the free energy $F = F(T,V)$ has to be minimized as a function of any additional parameters, such as $N$, requiring the condition $\frac{\partial F}{\partial N} = 0 = \mu$, which is the definition of chemical potential. The DOS is found in the usual manner (using the factor of 2 arising from two different polarizations of light):
$$2V \frac{d^3k}{(2\pi)^3} = D(\varepsilon)d\varepsilon. \quad (15)$$

Using the isotropic dispersion $\varepsilon_k = \hbar ck$, we obtain
$$2V \frac{d^3k}{(2\pi)^3} = \frac{V}{\pi^2} k^2 dk = \frac{\varepsilon^2}{\pi^2\hbar^3c^3} d\varepsilon, D(\varepsilon) = \frac{V\varepsilon^2}{\pi^2\hbar^3c^3}, D(\omega) = \frac{\omega^2}{\pi^2c^3}. \quad (16)$$

One way we can find the occupancy at each frequency $\omega$ is by writing down the grand potential for bosons:
$$-\beta \Omega(T,V,\mu) = -\sum_k \ln \left( 1 - \exp(-\beta(\varepsilon_k - \mu)) \right). \quad (17)$$

We can convert the sum over $k$ to an integral over $\varepsilon$ by using the density of states:
$$\Omega(T,V,\mu) = \int_0^\infty d\varepsilon D(\varepsilon) \ln(1 - e^{-\beta(\hbar\omega - \mu)}) \quad (18)$$
Differentiating this expression with respect to $\mu$ gives the total particle number:

$$N = -\frac{\partial \Omega}{\partial \mu} = \int_{0}^{\infty} d\omega \frac{D(\omega)}{e^{\beta \hbar \omega} - 1},$$

where we recognize $n(\omega) = \frac{1}{e^{\beta \hbar \omega} - 1}$ as the filling of each individual state by picking a specific frequency $\omega$.

**Problem 6**

The pressure of a degenerate, relativistic Fermi gas in three dimensions is (by analogy to radiation pressure) $P = \frac{1}{3} \rho \varepsilon_F$, using the density $\rho = \frac{N}{V}$. The Fermi momentum of a system of relativistic particles is found by the usual method:

$$N = 2V \int_{|k|<k_F} \frac{d^3k}{(2\pi)^3} = \frac{Vk_F^3}{3\pi^2}.$$  

The Fermi energy is related to the Fermi momentum as $\varepsilon_F = \hbar c k_F$ from the electron dispersion, so $P = \frac{\hbar c}{3} (3\pi^2)^{1/3} \rho^{4/3}$, where we allow the mass density to vary with position $\rho = \rho(r)$. Next, we find the mass of the dwarf enclosed by radius $r$

$$M(r) = \int d\Omega \int_0^r dr m_H r^2 \rho(r),$$

$m_H$ being the mass of the hydrogen atom. This gives us all of the terms in the expression for hydrostatic equilibrium of the star. The unusual feature of this solution is the pressure dependence $P \sim \rho^{4/3}$, unlike the usual $P \sim \rho^{5/3}$ dependence in the non-relativistic case.