May 23, Lecture 18

Last time: Magnetic Systems

Heisenberg model: \( \mathcal{H} = -2J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j - \sum_i \vec{S}_i \cdot \vec{H} \)

\[ \uparrow \text{nearest neighbor exchange integral} \]

Simplification: \( \vec{S}_i \cdot \vec{S}_j = S_i \sigma_j \sigma_j \) (uniaxial ferromagnet)

\( \rightarrow \) Ising model: \( \mathcal{H} = -J \sum_{i,j} \sigma_i \sigma_j - H \sum_i \sigma_i \)

\[ \sigma_i \in \{ \pm 1 \} \]

\( \sum_{\sigma_i} \mathcal{H}_i = -2J \)

Partition function: \( Z = \sum_{\sigma_i} e^{\beta \sum_{\sigma_i} \sigma_i} \)

\[ E = \langle \mathcal{H} \rangle = -\frac{2}{\beta} \log Z, \quad C_M = \left. \frac{\partial^2 \log Z}{\partial H^2} \right|_T \]

Avg magnetization: \( M = \langle \sum_i \sigma_i \rangle = \frac{1}{\beta} \frac{\partial}{\partial H} \log Z \)

Mag. susceptibility: \( \chi = \left. \frac{\partial M}{\partial H} \right|_{H \to 0} = \beta \left( \overline{M^2} - \overline{M}^2 \right) \)
Weiss Mean Field Theory

\[ H = -J \sum_i \sigma_i \sigma_j \]  
\[ \text{MF} \]

\[ \sigma \text{ sees the } \langle \sigma \rangle \text{ field } \]

of neighboring spin

\[ H_{\text{MF}} = -J \sum \sigma \sigma_i H \sigma_i \]

\[ \sum \sigma \right\}

\[ \mathbb{Z} = \sum_{\sigma} e^{-\beta H_{\text{MF}}} = \prod \left[ \sum_{\sigma_i = \pm 1} e^{-\beta \mathbb{H}(\sigma_i; \bar{\sigma})} \right] \]

value of \( \bar{\sigma} \)?

\[ \bar{\sigma}_i = \frac{\sum_{\sigma_i = \pm 1} \sigma_i e^{-\beta \mathbb{H}(\sigma_i; \bar{\sigma})}}{\sum_{\sigma_i = \pm 1} e^{-\beta \mathbb{H}(\sigma_i; \bar{\sigma})}} \]

\[ \bar{\sigma} = \frac{e^{\beta g \bar{\sigma} + H} - e^{-(\beta g \bar{\sigma} + H)}}{e^{\beta g \bar{\sigma} + H} + e^{-(\beta g \bar{\sigma} + H)}} \]

\[ \bar{\sigma} = \tanh(\beta g \bar{\sigma} + H) \]

\( \Rightarrow \) self-consistent relation for \( \bar{\sigma} \) (H, T)
let $J = g \beta J; \ h = \beta H$

$\bar{\sigma} = \tanh (g \bar{\sigma} + h)$

$h = 0$

$\begin{align*}
\bar{\sigma} & = 0 \\
\begin{array}{c|c}
J & \text{tanh}(g \bar{\sigma}) \\
\hline
< 1 & \uparrow \\
\geq 1 & \downarrow
\end{array}
\end{align*}$

Critical pt: $J_c = 1, T_c = \frac{8J}{k_B}$

$T > T_c, \bar{\sigma} = 0$ (paramagnet)

$T < T_c, \bar{\sigma} \neq 0$ (ferromagnet)

$\uparrow$ Spontaneous Symmetry breaking

C) Critical phenomenon

Vicinity of $T_c$: let $T = T_c \cdot (1 + t), \ |t| < 1$

$J = \frac{8J}{k_B T} \approx 1 - t$

expect $|\bar{\sigma}| \ll 1$ in the vicinity of $T_c$

$tanh (g \bar{\sigma}) \approx g \bar{\sigma} - \frac{(g \bar{\sigma})^3}{3}$

$\approx (1-t)\bar{\sigma} - \frac{(1-t)^3}{3} \bar{\sigma}^3$
Self-consistent eqn: \[ \dot{\sigma} = \left(1-t\right)\bar{\sigma} - \left(1-t\right)^2 \bar{\sigma}^3 / 3 \]

\[ \rightarrow t \bar{\sigma} = -\left(1-t\right) \bar{\sigma}^3 / 3 ; \]

\[ \text{Soln: } \bar{\sigma} = 0 \text{ or } \bar{\sigma}^2 = -3t / \left(1-t^2 \right) \approx -3t \]

\[ t > 0, \bar{\sigma} = 0 \text{ is only soln} \]

\[ t < 0, \bar{\sigma} = 0, \pm \sqrt{-3t} \]

\[ \rightarrow \text{to see which soln selected, compute free energy} \]

\[ -\beta \tilde{\Omega} = \delta \bar{\sigma} \sigma \quad (h = 0) \]

\[ Z = \left[ \sum_{\sigma=\pm 1} e^{-\beta \sum} \right]^N = \left[ e^{\sigma} + e^{-\sigma} \right]^N \]

\[ F = -k_B T \log Z = -N k_B T \log [2 \cosh \beta \bar{\sigma}] \]

\[ \text{for } t < 0, \text{ if } \bar{\sigma} = 0, F = -N k_B T \log 2 \equiv F_0 \]

\[ \text{if } \bar{\sigma} \neq 0, F = -N k_B T \log \left[ 2 \left( 1 + \frac{\bar{\sigma}^2}{2} \right) \right] < F_0 \]

\[ \Rightarrow \bar{\sigma} = \pm \sqrt{-3t} \alpha \left( T_c - T \right)^\beta \]

Phase coexistence for \( T < T_c \)

\[ \bar{\sigma}: \text{"Order parameter" of transition} \]

(no order at high temp; increasing order at low T)

\[ \text{Generally, } \bar{\sigma} \propto \left| T_c - T \right|^\beta, \text{ here } \beta = \frac{1}{2} \]
Energy:
\[ \langle H \rangle = \frac{M^2}{A} \langle \hat{H} \rangle = g J \sigma \cdot \langle \sigma \rangle \]
\[ = -g J \sigma^2 = \sum_q \gamma J \tau \quad t < 0 \]
\[ C_H = \frac{2}{A} \langle H \rangle = \sum_q \frac{3 \gamma J}{t} \quad t > 0 \]

- Discontinuous jump (but finite)
- Overall background temp dependence

\[ C_H \propto (T - T_c)^{-\alpha} \rightarrow \Box_{MF} \]

* Effect of magnetic field: Consider \( h > 0 \)
\[ \sigma = \tanh \left( J \overline{\sigma} + h \right) \]

\[ T > T_c \]

\[ T < T_c \]

Lowest free energy (largest \( \sigma^2 \))

Not selected
\[ f(n, h < 0, \sigma) \rightarrow -\sigma. \]

1st order transition across \( h = 0 \). \( f(n) \sim T < T_c \)

\[
\begin{array}{c}
\text{phase diagram:} \\
\begin{array}{c}
\sigma > 0 \\
\sigma < 0
\end{array}
\end{array}
\]

2nd order transition

phase coexistence

\[ \text{Cf. Liquid-gas transition} \]

\[ P \]

\[ \frac{P}{P_c} \]

\[ \begin{array}{c}
\text{gas} \\
\text{liquid}
\end{array} \]

\[ \frac{T}{T_c} \]

order parameter: \( \sigma_{\text{gas}} - \sigma_{\text{liq}} \)

Small field \( h \) around crit. pt.:

\[
\bar{\sigma} = \tanh(g \sigma + h) = (1-t)\bar{\sigma} - \frac{\bar{\sigma}^3}{3} + h \quad (\text{eqn of 566})
\]

\[ \uparrow \] Taylor expand again (expecting small change near \( T_c \))

\[
t\bar{\sigma} = -\frac{\bar{\sigma}^3}{3} + h \quad (\text{small } \bar{\sigma})
\]

\[ t > 0, \quad \bar{\sigma} \approx h/t
\]

\[ t = 0: \quad \bar{\sigma} = B h^{1/3}
\]

\[ \delta h^3 \]
Magnetic susceptibility:

\[ \chi = \frac{2\mu M}{dH} \propto N \frac{d\sigma}{dh} \quad \text{for} \quad T > T_c \]

To find \( \chi \) for \( T < T_c \), take \( \frac{\partial}{\partial h} \) of eqn of state:

\[ t\sigma = -\frac{\sigma^3}{3} + h \]

\[ t \frac{\partial \sigma}{\partial h} + \sigma^2 \frac{\partial \sigma}{\partial h} = 1 \]

\[ \chi \propto \frac{\sigma^2}{T - \sigma^2} \]

\[ \chi \propto \begin{cases} \frac{1}{T - \sigma^2} & t > 0 \\ -\frac{2t}{T - \sigma^2} & t < 0 \end{cases} \]

Generally, \( \chi \propto |T - T_c|^{-\delta} \rightarrow [\delta_{\text{MF}} = 1] \)

Note: exponents \( \alpha, \beta, \delta, \sigma \) same as liq-gas transition

\( \Rightarrow \) same "universality class"